A Possible Boson Realization of Generalized Lipkin Model for Many-Fermion System

—— The su(M+1)-Algebraic Model in Non-Symmetric Boson Representation ——

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Abstract

On the basis of the formalism proposed by three of the present authors (A. K., J. P. & M. Y.), generalized Lipkin model consisting of $(M+1)$ single-particle levels is investigated. This model is essentially a kind of the $su(M+1)$-algebraic model and, in contrast to the conventional treatment, the case, where fermions are partially occupied in each level, is discussed. The scheme for obtaining the orthogonal set for the irreducible representation is presented.

§1. Introduction and preliminaries

The Lipkin model,[1] which was proposed by Lipkin, Meshkov and Glick in 1965, has played a peculiar role in schematic studies of collective motions in many-fermion system. This is a kind of shell model : Under certain interaction, many fermions move in two single-particle levels with the same degeneracy. This is essentially one kind of the $su(2)$-algebraic model and it has also contributed to schematic understanding of finite temperature effects in many-fermion system.[2] On the other hand, the Lipkin model has been generalized to the case of many single-particle levels, for example, the case of three levels is the most popular and it is a kind of the $su(3)$-algebraic model. The investigations based on this model is also not only for collective motion[3] but also for finite temperature effects in many-fermion system.[4]

The generators of the $su(2)$ algebra, which we denote $(\widetilde{S}_+, \widetilde{S}_-, \widetilde{S}_0)$, play a central role in the Lipkin model. These are expressed in terms of the bilinear forms for particle operators $(\widetilde{\alpha}_{1m}, \widetilde{\alpha}_{1m}^*)$ and hole operators $(\widetilde{\beta}_{\widetilde{m}}, \widetilde{\beta}_{\widetilde{m}}^*)$:

$$\widetilde{S}_+ = \sum_m \widetilde{\alpha}_{1m}^* \widetilde{\beta}_{\widetilde{m}}^* \ , \qquad \widetilde{S}_- = \sum_m \widetilde{\beta}_{\widetilde{m}} \widetilde{\alpha}_{1m} \ ,$$
$$\widetilde{S}_0 = \frac{1}{2}\left[\sum_m \widetilde{\alpha}_{1m}^* \widetilde{\alpha}_{1m} - (\Omega - \sum_m \widetilde{\beta}_{\widetilde{m}}^* \widetilde{\beta}_{\widetilde{m}})\right] \ . \qquad (1\cdot1a)$$

Here, Ω denotes the degeneracy of the single-particle levels. The definitions of the other notations will be given in §2. For the later convenience, we will use $(\widetilde{S}^1, \widetilde{S}_1, \widetilde{S}_1^1)$ for the $su(2)$ generators defined as

$$\widetilde{S}^1 = \widetilde{S}_+ \ , \quad \widetilde{S}_1 = \widetilde{S}_- \ , \quad \widetilde{S}_1^1 = 2\widetilde{S}_0 \ . \qquad (1\cdot1b)$$

In addition to the above generators, there exists one operator, which we denote $\widetilde{N}$ as

$$\widetilde{N} = (\Omega - \sum_m \widetilde{\beta}_{\widetilde{m}}^* \widetilde{\beta}_{\widetilde{m}}) + \sum_m \widetilde{\alpha}_{1m}^* \widetilde{\alpha}_{1m} \ . \qquad (1\cdot2)$$

The operator $\widetilde{N}$ denotes the total fermion number and it obeys

$$\left[\widetilde{N}, \widetilde{S}^1\right] = \left[\widetilde{N}, \widetilde{S}_1\right] = \left[\widetilde{N}, \widetilde{S}_1^1\right] = 0 \ . \qquad (1\cdot3)$$

Conventionally, the orthogonal set for the Lipkin model is obtained by operating $\widetilde{S}^1$ successively on the state $|m\rangle\rangle$ which obeys the condition

$$\widetilde{S}_1 |m\rangle\rangle = 0 \ , \qquad \widetilde{S}_1^1 |m\rangle\rangle = -\sigma_1 |m\rangle\rangle \ . \qquad (1\cdot4)$$

The state $|m\rangle\rangle$ is the eigenstate of the Casimir operator of the $su(2)$ algebra :

$$\widetilde{\Gamma}_{su(2)} |m\rangle\rangle = \frac{1}{2}\sigma_1(\sigma_1 + 2) |m\rangle\rangle \ ,$$
$$\widetilde{\Gamma}_{su(2)} = \widetilde{S}^1 \widetilde{S}_1 + \widetilde{S}_1 \widetilde{S}^1 + \frac{1}{2}\left(\widetilde{S}_1^1\right)^2 \ . \qquad (1\cdot5)$$

The quantity $\sigma_1/2$ denotes the magnitude of the quasi-spin. Further, the relation (1·3) supports that $|m\rangle\rangle$ is also the eigenstate of $\widetilde{N}$:

$$\widetilde{N}|m\rangle\rangle = N|m\rangle\rangle \ . \tag{1·6}$$

By operating $\widetilde{S}^1$ for $(\sigma_1+\sigma_0)/2$ times on $|m\rangle\rangle$, we have

$$\begin{aligned} &|(\gamma);N,\sigma_1,\sigma_0\rangle\rangle = \left(\widetilde{S}^1\right)^{(\sigma_1+\sigma_0)/2}|m\rangle\rangle \ , \\ &|m\rangle\rangle = |(\gamma);N,\sigma_1\rangle\rangle \ . \end{aligned} \tag{1·7}$$

Here, (γ) denotes a set of the quantum numbers additional to those related to the $su(2)$ algebra. The definitions of $\widetilde{S}^1_1$ and $\widetilde{N}$ tell us that $|m\rangle\rangle$ can be specified by the quantum numbers n and n_1 which are the eigenvalues of $\sum_m \widetilde{\beta}^*_{\widetilde{m}}\widetilde{\beta}_{\widetilde{m}}$ and $\sum_m \widetilde{\alpha}^*_{1m}\widetilde{\alpha}_{1m}$, respectively :

$$|m\rangle\rangle = |(\gamma);n,n_1\rangle\rangle \ . \tag{1·8}$$

The state $|m\rangle\rangle$ is called the minimum weightly state, but, in this paper, we will call it the intrinsic state in the meaning analogous to the rotational model. The relation between (N,σ_1) and (n,n_1) is given as

$$N = \mathit{\Omega} - n + n_1 \ , \qquad \sigma_1 = \mathit{\Omega} - n - n_1 \ , \tag{1·9a}$$

$$n = \mathit{\Omega} - \frac{1}{2}(N+\sigma_1) \ , \qquad n_1 = \frac{1}{2}(N-\sigma_1) \ . \tag{1·9b}$$

Since $0 \le \mathit{\Omega} - n \le \mathit{\Omega}$ and $0 \le n_1 \le \mathit{\Omega}$, we have the following relations :

$$\text{if} \ \ 0 \le N \le \mathit{\Omega} \ , \qquad 0 \le \sigma_1 \le N \ , \tag{1·10a}$$

$$\text{if} \ \ \mathit{\Omega} \le N \le 2\mathit{\Omega} \ , \qquad 0 \le \sigma_1 \le 2\mathit{\Omega} - N \ , \tag{1·10b}$$

The above is an outline of the Lipkin model and it is easily generalized to the $su(M+1)$-algebraic model in the $(M+1)$ single-particle levels with the same degeneracy, which will be discussed in §3.

Our present interest is concerned with the explicit determination of $|m\rangle\rangle$ in terms of the fermion operators. With the use of the quasi-fermion operators obeying certain constraints, [5)] but, it may be difficult to apply this idea to the generalized case given in §3. However, one case is simply given : $n = n_1 = 0$, i.e., $N = \sigma_1 = \mathit{\Omega}$. In this case, $|m\rangle\rangle$ corresponds to the vacuum of $\widetilde{\alpha}_{1m}$ and $\widetilde{\beta}_{\widetilde{m}}$ $(\widetilde{\alpha}_{1m}|m\rangle\rangle = \widetilde{\beta}_{\widetilde{m}}|m\rangle\rangle = 0)$ and it means that one level is fully occupied by the fermions, i.e., the closed shell system. The case of the generalized Lipkin model is in the same situation as that in the $su(2)$ model. Except the case of the finite temperature effects, [2)] many of the investigations based on the Lipkin model is restricted to the case $n = n_1 = 0$, that is, the use of the orthogonal set obtained by operating $\widetilde{S}^1$ on the state $|(\gamma); n = 0, n_1 = 0\rangle\rangle$ $(= |(\gamma); N = \Omega, \sigma_1 = \Omega\rangle\rangle)$. In Ref.2), all the values of σ_1 permitted, which are shown in the relation (1·10a), are taken into account for the system with $N = \Omega$. The case of the $su(3)$ Lipkin model[3)] is also in the situation similar to the above except the finite temperature effects.[4)]

As is well known, there exist two forms for the boson realization of the Lipkin model. One is the Holstein-Primakoff representation[6)] and the other the Schwinger representation.[7)] In the former, the three generators are expressed as

$$\widetilde{\mathcal{S}}^1 = \widetilde{C}^* \cdot \sqrt{S_1 - \widetilde{C}^*\widetilde{C}} \ , \qquad \widetilde{\mathcal{S}}_1 = \sqrt{S_1 - \widetilde{C}^*\widetilde{C}} \cdot \widetilde{C} \ ,$$
$$\widetilde{\mathcal{S}}_1^1 = 2\widetilde{C}^*\widetilde{C} - S_1 \ . \tag{1·11}$$

Here, $(\widetilde{C}, \widetilde{C}^*)$ denote boson operators and σ_1 is given in the relation (1·5). The state $|s_1, s_0)$ is constructed in the form

$$|s_1, s_0) = (\widetilde{\mathcal{S}}^1)^{(s_1+s_0)/2}|m) = (\widetilde{C}^*)^{(s_1+s_0)/2}|0) \ ,$$
$$|m) = |0) \ . \quad (\widetilde{C}|0) = 0) \tag{1·12}$$

The Schwinger representation gives the following form for the generators :

$$\hat{S}^1 = \hat{a}_1^*\hat{b} \ , \qquad \hat{S}_1 = \hat{b}^*\hat{a}_1 \ , \qquad \hat{S}_1^1 = \hat{a}_1^*\hat{a}_1 - \hat{b}^*\hat{b} \ . \tag{1·13}$$

Here, $(\hat{a}_1, \hat{a}_1^*)$ and $(\hat{b}, \hat{b}^*)$ denote two kinds of bosons. The state $|s_1, s_0\rangle$ is constructed in the form

$$|s_1, s_0\rangle = (\hat{S}^1)^{(s_1+s_0)/2}|m\rangle = (\hat{a}_1^*)^{(s_1+s_0)/2}(\hat{b}^*)^{(s_1-s_0)/2}|0\rangle \ ,$$
$$|m\rangle = (\hat{b}^*)^{s_1}|0\rangle \ . \quad (\hat{a}_1|0\rangle = \hat{b}|0\rangle = 0) \tag{1·14}$$

In a certain case mentioned below, the Holstein-Primakoff representation is easily generalized and it is called the symmetric representation.[8)]

The above two boson realizations have undoubtedly contributed in the development of the studies of many-fermion systems. In order to connect the above boson systems with the original fermion system, it may be natural to set up the following condition :

$$s_1 = \sigma_1 \ (= \Omega - n - n_1 = 2(\Omega - n) - N) \ . \tag{1·15}$$

The relation (1·15) comes from the relation (1·9a). Of course, the quantity s_1 obeys the restrictions (1·10). However, usually, the above two boson realizations have been applied to the case where one level is fully occupied : $s_1 = N = \Omega$, i.e., $n = n_1 = 0$. This means that, with the help of these forms, we are able to obtain only schematic knowledges on the low-lying states of so-called closed shell system. In other words, it may be impossible to get any information not only on the low-lying states of open shell systems but also on the high-lying states of closed or open shell systems. Further, as was already mentioned, the Holstein-Primakoff representation was generalized in the frame of the symmetric representation, which enable us to describe the closed shell systems. If we intend to investigate the above-mentioned cases in the frame of the boson space, we must generalize the boson realization of the Lipkin model in the form containing the condition (1·15), i.e., in the form of non-symmetric representation.

Main aim of this paper is to present a possible boson realization of generalized Lipkin model in non-symmetric representation. We investigate a model consisting of $(M+1)$ single-particle levels with the same degeneracy. This is nothing but generalized Lipkin model. [8] In the intrinsic state for this model, each single-particle level is partially occupied by the fermions. This is the most important point of the present investigation. For this investigation, a general framework for the $su(M+1)$ algebra in the Schwinger boson representation may be useful. This framework was proposed by three of the present authors (A. K., J. P. & M. Y.), which will be, hereafter, referred to as (I). [9] In (I), $(M+1)(N+1)$ kinds of boson operators are prepared. With the use of them, we can construct $((M+1)^2-1)$ generators of the $su(M+1)$ algebra in terms of the bilinear forms for the bosons. On the other hand, the $su(N,1)$ algebra, which is independent of the above $su(M+1)$ algebra, is constructed in the frame of the above bosons. Further, it can be seen that there exists one operator, which commutes with all the generators of both algebras. This operator plays a role similar to that of the fermion number operator in the Lipkin model. With the aid of this operator, we can formulate the $su(M+1)$ and the $su(N,1)$ algebraic model and under certain correspondence between the fermion and the boson space, we are able to have the generalized Lipkin model in the framework of the Schwinger boson representation developed in (I). The conventional form is called the symmetric boson representation and the present one may be called non-symmetric boson representation. Especially, the intrinsic state with the partially occupied single-particle levels can be expressed explicitly in terms of the boson operators. Under the general scheme for obtaining the orthogonal set, we can show two cases, the $su(2)$ and the $su(3)$ algebra in the explicit form.

In next section, we will recapitulate the essential part of (I) for the present discussion. Section 3 will be devoted to giving the generalized Lipkin model in the fermion space. In §4, we will discuss the method how to construct the intrinsic state in the boson space. Through this discussion, it is enough for the present aim to consider the case $M=N$ for the $su(M+1)$ and the $su(N,1)$ algebra. In §5, the correspondence between the intrinsic states for the fermion and the boson space will be discussed. Section 6 will be devoted to giving the general scheme for constructing the orthogonal set for the irreducible representation. Finally, in §7, the cases of the $su(2)$ and the $su(3)$ algebras will be explicitly presented and the concluding remarks will be mentioned. In Appendices, some mathematical formulae and proof, which are needed in this paper, will be given.

§2. A possible Schwinger boson representation for the $su(M+1)$ algebra and its related $su(N,1)$ algebra

In (I), we developed a possible boson representation for the $su(M+1)$ algebra, which is a natural extension of the Schwinger boson representation for the $su(2)$ algebra. First, we recapitulate its representation. Let us introduce a boson space which is constructed in terms of $(M+1)(N+1)$ kinds of boson operators : $(\hat{a}_i, \hat{a}_i^*)$, $(\hat{a}^p, \hat{a}^{p*})$, $(\hat{b}_i^p, \hat{b}_i^{p*})$ and $(\hat{b}, \hat{b}^*)$, where $i = 1, 2, \cdots, M$ and $p = 1, 2, \cdots, N$. In this space, the following bilinear forms are defined :

$$\hat{S}^i = \hat{a}_i^* \hat{b} + \sum_{p=1}^{N} \hat{a}^{p*} \hat{b}_i^p \ , \qquad \hat{S}_i = \hat{b}^* \hat{a}_i + \sum_{p=1}^{N} \hat{b}_i^{p*} \hat{a}^p \ , \tag{2·1a}$$

$$\hat{S}_i^j = \hat{a}_i^* \hat{a}_j - \sum_{p=1}^{N} \hat{b}_j^{p*} \hat{b}_i^p + \delta_{ij} \left(\sum_{p=1}^{N} \hat{a}^{p*} \hat{a}^p - \hat{b}^* \hat{b} \right) \ . \tag{2·1b}$$

The set $(\hat{S}^i, \hat{S}_i, \hat{S}_i^j)$ composes the $su(M+1)$ algebra :

$$\hat{S}_i^* = \hat{S}^i \ , \qquad \hat{S}_j^{i*} = \hat{S}_i^j \ , \tag{2·2a}$$

$$\begin{aligned} &[\hat{S}^i, \hat{S}^j] = 0 \ , \qquad [\hat{S}^i, \hat{S}_j] = \hat{S}_i^j \ , \\ &[\hat{S}_i^j, \hat{S}^k] = \delta_{jk} \hat{S}^i + \delta_{ij} \hat{S}^k \ , \\ &[\hat{S}_i^j, \hat{S}_k^l] = \delta_{jk} \hat{S}_i^l - \delta_{il} \hat{S}_k^j \ . \end{aligned} \tag{2·2b}$$

In associating with the above set, we introduce an operator $\hat{S}$ in the form

$$\hat{S} = \sum_{i=1}^{M} \hat{a}_i^* \hat{a}_i - \sum_{p=1}^{N} \hat{a}^{p*} \hat{a}^p - \sum_{i=1}^{M} \sum_{p=1}^{N} \hat{b}_i^{p*} \hat{b}_i^p + \hat{b}^* \hat{b} \ . \tag{2·3}$$

The operator $\hat{S}$ cannot be expressed in terms of any function for $(\hat{S}^i, \hat{S}_i, \hat{S}_i^j)$ and satisfies

$$[\hat{S}, \hat{S}^i] = [\hat{S}, \hat{S}_i] = [\hat{S}, \hat{S}_i^j] = 0 \ . \tag{2·4}$$

The Casimir operator $\hat{\Gamma}_{su(M+1)}$, which commutes with $(\hat{S}^i, \hat{S}_i, \hat{S}_i^j)$, is expressed as follows :

$$\hat{\Gamma}_{su(M+1)} = \sum_{i=1}^{M} (\hat{S}^i \hat{S}_i + \hat{S}_i \hat{S}^i) + \sum_{i,j=1}^{M} \hat{S}_j^i \hat{S}_i^j - (M+1)^{-1} \left(\sum_{i=1}^{M} \hat{S}_i^i \right)^2$$

$$= 2\left(\sum_{i=1}^{M}\hat{S}^i\hat{S}_i + \sum_{j>i}\hat{S}_j^i\hat{S}_i^j\right)$$

$$+\sum_{i=1}^{M}(\hat{S}_i^i)^2 - (M+1)^{-1}\left(\sum_{i=1}^{M}\hat{S}_i^i\right)^2 + \sum_{i=1}^{M}(M-2i)\hat{S}_i^i\ . \qquad (2\cdot5)$$

In the present boson space, we can define another set of the bilinear forms :

$$\hat{T}^p = \hat{a}^{p*}\hat{b}^* - \sum_{i=1}^{M}\hat{a}_i^*\hat{b}_i^{p*}\ , \qquad \hat{T}_p = \hat{b}\hat{a}^p - \sum_{i=1}^{M}\hat{b}_i^p\hat{a}_i\ , \qquad (2\cdot6\text{a})$$

$$\hat{T}_q^p = \hat{a}^{p*}\hat{a}^q + \sum_{i=1}^{M}\hat{b}_i^{p*}\hat{b}_i^q + \delta_{pq}\left(\sum_{i=1}^{M}\hat{a}_i^*\hat{a}_i + \hat{b}^*\hat{b} + (M+1)\right)\ . \qquad (2\cdot6\text{b})$$

The set $(\hat{T}^p, \hat{T}_p, \hat{T}_q^p)$ composes the $su(N,1)$ algebra and satisfies the following relations :

$$\hat{T}_p^* = \hat{T}^p\ , \qquad \hat{T}_p^{q*} = \hat{T}_q^p\ , \qquad (2\cdot7\text{a})$$

$$[\hat{T}^p, \hat{T}^q] = 0, \qquad [\hat{T}^p, \hat{T}_q] = -\hat{T}_q^p\ ,$$

$$[\hat{T}_q^p, \hat{T}^r] = \delta_{qr}\hat{T}^p + \delta_{pq}\hat{T}^r\ ,$$

$$[\hat{T}_q^p, \hat{T}_s^r] = \delta_{qr}\hat{T}_s^p - \delta_{ps}\hat{T}_q^r\ . \qquad (2\cdot7\text{b})$$

We can easily verify the relation

$$[\hat{S}, \hat{T}^p] = [\hat{S}, \hat{T}_p] = [\hat{S}, \hat{T}_q^p] = 0\ . \qquad (2\cdot8)$$

The Casimir operator $\hat{\Gamma}_{su(N,1)}$, which commutes with $(\hat{T}^p, \hat{T}_p, \hat{T}_q^p)$, is expressed as follows :

$$\hat{\Gamma}_{su(N,1)} = -\sum_{p=1}^{N}(\hat{T}^p\hat{T}_p + \hat{T}_p\hat{T}^p) + \sum_{p,q=1}^{N}\hat{T}_q^p\hat{T}_p^q - (N+1)^{-1}\left(\sum_{p=1}^{N}\hat{T}_p^p\right)^2$$

$$= -2\left(\sum_{p=1}^{N}\hat{T}^p\hat{T}_p - \sum_{q>p}\hat{T}_p^q\hat{T}_q^p\right)$$

$$+\sum_{p=1}^{N}(\hat{T}_p^p)^2 - (N+1)^{-1}\left(\sum_{p=1}^{N}\hat{T}_p^p\right)^2 + \sum_{p=1}^{N}(N-2p)\hat{T}_p^p\ . \qquad (2\cdot9)$$

The above is the recapitulation of (I) and, finally, a very important fact should be mentioned. The two sets $(\hat{S}^i, \hat{S}_i, \hat{S}_i^j)$ and $(\hat{T}^p, \hat{T}_p, \hat{T}_q^p)$ commute mutually for any components :

$$[\hat{S}^i, \hat{T}^p] = 0\ , \quad \text{etc.} \qquad (2\cdot10)$$

§3. A generalized Lipkin model

We consider a nuclear shell model consisting of $(M+1)$ single-particle levels, each of which is specified by a quantum number i $(i = 0, 1, 2, \cdots, M)$. The degeneracy of each level is Ω-fold. Then, introducing a quantum number m $(m = -j, -j+1, \cdots, j-1, j \ : j =$ half-integer, $\Omega = 2j+1)$, the single-particle state can be specified by a set of the quantum numbers (i, m). In other places, we use the notations j and m in a meaning different from the above, but, the confusion may not occur. For the fermion operators, we use particle creation and annihilation operator in the state $(i, m : i = 1, 2, \cdots, M)$ by $\widetilde{\alpha}_{im}^*$ and $\widetilde{\alpha}_{im}$, respectively. For the state $(i = 0, m)$, we use hole creation and annihilation operator $\widetilde{\beta}_{\widetilde{m}}^*$ and $\widetilde{\beta}_{\widetilde{m}}$, respectively, where the symbol $\widetilde{m}$ is used for abbreviating $(-1)^{j-m}\widetilde{T}_{-m} = \widetilde{T}_{\widetilde{m}}$.

With the use of the above fermion operators, $\widetilde{\alpha}_{im}^*$, $\widetilde{\alpha}_{im}$, $\widetilde{\beta}_{\widetilde{m}}^*$ and $\widetilde{\beta}_{\widetilde{m}}$, we define the following bilinear form :

$$\widetilde{S}^i = \sum_m \widetilde{\alpha}_{im}^* \widetilde{\beta}_{\widetilde{m}}^* \ , \qquad \widetilde{S}_i = \sum_m \widetilde{\beta}_{\widetilde{m}} \widetilde{\alpha}_{im} \ , \tag{3·1a}$$

$$\widetilde{S}_i^j = \sum_m \widetilde{\alpha}_{im}^* \widetilde{\alpha}_{jm} - \delta_{ij} \left(\Omega - \sum_m \widetilde{\beta}_{\widetilde{m}}^* \widetilde{\beta}_{\widetilde{m}} \right) \ . \tag{3·1b}$$

The symbol j in the form (3·1b) is used for specifying the j-th single-particle level. We know that the above set $(\widetilde{S}^i, \widetilde{S}_i, \widetilde{S}_i^j)$ composes the $su(M+1)$ algebra and the properties are the same as those given in the relations (2·2a) and (2·2b). The Casimir operator $\widetilde{\Gamma}_{su(M+1)}$ is also of the same form as that given in the relation (2·5). In addition to the above mathematical framework, conventionally, the simplest case, where there are Ω fermions, exactly the degeneracy of each level, has been investigated. Certainly, the case $M = 1$ corresponds to the Lipkin model. In this sense, the above may be called as a generalized Lipkin model.

However, we should note that there exists another way for the generalization of the Lipkin model. As was already mentioned, the total fermion number in the Lipkin model is equal to the degeneracy of each level. Keeping this relation, the above-mentioned model is generalized from the $su(2)$ algebra to the $su(M+1)$ algebra. Therefore, another way for the generalization is found in the case where the total fermion number is different of the degeneracy of each level. Then, in associating with the set $(\widetilde{S}^i, \widetilde{S}_i, \widetilde{S}_i^j)$, we introduce an operator $\widetilde{N}$ in the form

$$\widetilde{N} = \left(\Omega - \sum_m \widetilde{\beta}_{\widetilde{m}}^* \widetilde{\beta}_{\widetilde{m}} \right) + \sum_{i=1}^{M} \left(\sum_m \widetilde{\alpha}_{im}^* \widetilde{\alpha}_{im} \right) \ . \tag{3·2}$$

The operator $\widetilde{N}$ represents the total fermion number and satisfies

$$[\widetilde{N}, \widetilde{S}^i] = [\widetilde{N}, \widetilde{S}_i] = [\widetilde{N}, \widetilde{S}_i^j] = 0 \ . \tag{3·3}$$

It should be also noted that $\widetilde{N}$ cannot be expressed in terms of any function for $(\widetilde{S}^i, \widetilde{S}_i, \widetilde{S}_i^j)$. In this paper, we will investigate the case where the total fermion number is different of the degeneracy of each level, even if the case of the $su(2)$ algebra.

Let us presuppose that, in the present fermion space, there exists the state $|m\rangle\rangle$ uniquely, which obeys the following conditions :

$$\widetilde{S}_i|m\rangle\rangle = 0 \ , \qquad \widetilde{S}_i^j|m\rangle\rangle = 0 \ . \quad (j > i) \tag{3·4}$$

Then, operating $\widetilde{S}_i^i$ on the both sides of the conditions (3·4) and using the commutation relations, we have

$$\widetilde{S}_i \cdot \widetilde{S}_i^i|m\rangle\rangle = 0 \ , \qquad \widetilde{S}_i^j \cdot \widetilde{S}_i^i|m\rangle\rangle = 0 \ . \quad (j > i) \tag{3·5}$$

Under the relations (3·5), the presupposition of the existence of the unique $|m\rangle\rangle$ leads us to

$$\widetilde{S}_i^i|m\rangle\rangle = -\sigma_i|m\rangle\rangle \ . \quad (i = 1, 2, \cdots, M) \tag{3·6}$$

Here, σ_i denotes c-number. The relation (3·6) is nothing but the eigenvalue equation for the operator $\widetilde{S}_i^i$ and σ_i is real. Further, operating $\widetilde{N}$ on the both sides of the conditions (3·4), together with the relation (3·6), we have

$$\sum_m \widetilde{\beta}_{\widetilde{m}}^* \widetilde{\beta}_{\widetilde{m}}|m\rangle\rangle = n|m\rangle\rangle \ , \tag{3·7a}$$

$$\sum_m \widetilde{\alpha}_{im}^* \widetilde{\alpha}_{im}|m\rangle\rangle = n_i|m\rangle\rangle \ , \tag{3·7b}$$

$$N = \Omega - n + \sum_{i=1}^{M} n_i \ , \qquad \Omega - n = (M+1)^{-1}\left(N + \sum_{j=1}^{M}\sigma_j\right) \ , \tag{3·8}$$

$$\sigma_i = \Omega - n - n_i \ , \qquad n_i = (M+1)^{-1}\left(N + \sum_{j=1}^{M}\sigma_j\right) - \sigma_i \ . \tag{3·9}$$

The relation (3·8) tells us that we are interested in a system with N $(= \Omega - n + \sum_{i=1}^{M} n_i)$ fermion number. The above consideration means that the state $|m\rangle\rangle$ is specified by

$$|m\rangle\rangle = |(\gamma); N, \sigma_1, \cdots, \sigma_M\rangle\rangle \quad \text{or} \quad |(\gamma); n, n_1, \cdots, n_M\rangle\rangle \ . \tag{3·10}$$

Here, (γ) denotes a set of the quantum numbers which do not relate directly to the present algebra.

Now, let us investigate properties of the quantum numbers specifying the state (3·10). First, we note the relations

$$\langle\langle m|\widetilde{S}_i \cdot \widetilde{S}^i|m\rangle\rangle \geq 0 \ , \qquad \langle\langle m|\widetilde{S}_i^j \cdot \widetilde{S}_j^i|m\rangle\rangle \geq 0 \ . \ (j > i) \tag{3·11}$$

The conditions (3·11) give us the restrictions

$$\sigma_i \geq 0\ , \quad \text{i.e.,} \quad n + n_i \leq \Omega\ , \tag{3·12}$$

$$\sigma_i \leq \sigma_j\ , \quad \text{i.e.,} \quad n_i \geq n_j\ . \ \ (j > i) \tag{3·13}$$

The restrictions (3·12) and (3·13) are summarized as

$$\Omega - n \geq n_1 \geq n_2 \geq \cdots \geq n_M\ . \tag{3·14}$$

It may be self-evident that the state $|m\rangle\rangle$ is the eigenstate of the Casimir operator $\widetilde{\Gamma}_{su(M+1)}$ with the eigenvalue $\gamma^{(f)}_{su(M+1)}$, which is given as

$$\gamma^{(f)}_{su(M+1)} = \sum_{i=1}^{M} \sigma_i^2 - (M+1)^{-1}\left(\sum_{i=1}^{M}\sigma_i\right)^2 - \sum_{i=1}^{M}(M-2i)\sigma_i\ . \tag{3·15}$$

The above form is obtained through the relation (2·5) replaced $\hat{S}^i_i$ with $\widetilde{S}^i_i$ and Eq.(3·6). If c, e, L, X_c, X^e and X^L in the relations (A·16) and (A·17) read i, k, M, $-\sigma_i$, $-\sigma^k$ and $-\sigma^M$, respectively, the form (3·15) can be also written as follows :

$$\gamma^{(f)}_{su(M+1)} = \sum_{k=1}^{M}[k/(k+1)]\sigma^k[\sigma^k + (k+1)]\ , \tag{3·16}$$

$$\sigma^k = \begin{cases} -(1/k)\sum_{i=1}^{k}\sigma_i + \sigma_{k+1}\ , & (k = 1, 2, \cdots, M-1) \\ (1/M)\sum_{i=1}^{M}\sigma_i\ . & (k = M) \end{cases} \tag{3·17}$$

Then, the state $|m\rangle\rangle$ can be also specified as

$$|m\rangle\rangle = |(\gamma); N, \sigma^1, \cdots, \sigma^M\rangle\rangle\ . \tag{3·18}$$

Under the restriction (3·13), we can show that $\gamma^{(f)}_{su(M+1)}$ is positive-definite.

Thus, we learned the properties of the state $|m\rangle\rangle$ which obeys the conditions (3·4). The state $|m\rangle\rangle$ is specified by $(M+1)$ quantum numbers. By regarding $|m\rangle\rangle$ as the intrinsic state and by operating $\widetilde{S}^i$ and $\widetilde{S}^i_j$ $(j > i)$ in an appropriate order on $|m\rangle\rangle$, we are able to obtain the orthogonal set for the $su(M+1)$ algebra. The number of $\widetilde{S}^i$ and $\widetilde{S}^i_j$ $(j > i)$ are M and $(M^2 - M)/2$, respectively and totally, $(M^2 + M)/2$. Then, the orthogonal set is specified by $(M+1)(M+2)/2$ $(= (M^2 + M)/2 + (M+1))$ quantum numbers.

§4. Construction of the intrinsic state in the boson space

Now, let us investigate the intrinsic state playing the same role as that of $|m\rangle\rangle$ shown in §3 in the boson space defined in §2. First, we impose the following conditions to the state

$|m\rangle$:

$$\hat{S}_i|m\rangle = 0\ , \qquad \hat{S}_i^j|m\rangle = 0\ , \quad (j > i) \tag{4·1a}$$

$$\hat{T}_p|m\rangle = 0\ , \qquad \hat{T}_q^p|m\rangle = 0\ . \quad (q > p) \tag{4·1b}$$

For a moment, the conditions (4·1a) and (4·1b) should be regarded as the supposition, which is in the same situation as that in §3. Further, let the state $|m\rangle$ contain $(M+1)$ quantum numbers. Then, by operating $\hat{S}^i$, $\hat{S}_j^i$ $(j > i)$, $\hat{T}^p$ and $\hat{T}_p^q$ $(q > p)$ appropriately on the state $|m\rangle$, we are able to obtain the orthogonal set for the present algebras. If it is possible, the total number of the quantum numbers is given by $(N^2+N)/2 + (M^2+M)/2 + (M+1)$. On the other hand, our present boson space consists of $(M+1)(N+1)$ kinds of bosons and, then, we have the relation

$$(N^2+N)/2 + (M^2+M)/2 + (M+1) = (M+1)(N+1)\ . \tag{4·2}$$

The above relation gives us

$$N = M\ , \qquad N = M+1\ . \tag{4·3}$$

In this paper, we will treat the case $N = M$.

In §3, starting from the presupposition of the existence of the unique $|m\rangle\rangle$, we showed its various properties, but, we did not prove the existence. In this section, in the same idea as that in §3, we will start with the presupposition of the unique $|m\rangle$, but, the proof of the existence and its explicit form are given. From the condition $\hat{S}_i|m\rangle = \hat{T}_p|m\rangle = 0$ for $i, p = 1, 2, \cdots, M$ shown in Eqs.(4·1a) and (4·1b), we can conclude that the intrinsic state $|m\rangle$ contains only $\hat{b}_i^{p*}$ $(i, p = 1, 2, \cdots, M)$ and $\hat{b}^*$. Therefore, it may be enough to consider the conditions $\hat{S}_i^j|m\rangle = \hat{T}_q^p|m\rangle = 0$ $(j > i, q > p)$. In order to obtain the state $|m\rangle$ which satisfies the above conditions, we introduce the following operator :

$$\hat{B}_r^* = \begin{vmatrix} \hat{b}_M^{1*} & \hat{b}_M^{2*} & \cdots & \hat{b}_M^{r*} \\ \hat{b}_{M-1}^{1*} & \hat{b}_{M-1}^{2*} & \cdots & \hat{b}_{M-1}^{r*} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{b}_{M-r+1}^{1*} & \hat{b}_{M-r+1}^{2*} & \cdots & \hat{b}_{M-r+1}^{r*} \end{vmatrix}\ . \quad (r = 1, 2, \cdots, M) \tag{4·4}$$

For example, $\hat{B}_1^*$ and $\hat{B}_2^*$ are given as

$$\hat{B}_1^* = \hat{b}_M^{1*}\ , \qquad \hat{B}_2^* = \begin{vmatrix} \hat{b}_M^{1*} & \hat{b}_M^{2*} \\ \hat{b}_{M-1}^{1*} & \hat{b}_{M-1}^{2*} \end{vmatrix} = \hat{b}_M^{1*}\hat{b}_{M-1}^{2*} - \hat{b}_{M-1}^{1*}\hat{b}_M^{2*}\ . \tag{4·5}$$

The operator $\hat{B}_r^*$ obeys the relations

$$[\hat{S}_i^j, \hat{B}_r^*] = 0\ , \quad (j > i) \tag{4·6a}$$

$$[\hat{T}_q^p, \hat{B}_r^*] = 0\ . \quad (q > p) \tag{4·6b}$$

The above relations are proved in Appendix B. As for a possible form for $|m\rangle$, we adopt the following form :

$$|m\rangle = (\hat{B}_1^*)^{\nu_M}(\hat{B}_2^*)^{\nu_{M-1}}\cdots(\hat{B}_r^*)^{\nu_{M-r+1}}\cdots(\hat{B}_M^*)^{\nu_1}(\hat{b}^*)^{\nu}|0\rangle \ . \tag{4·7}$$

Since $\hat{S}_i^j|0\rangle = \hat{T}_q^p|0\rangle = 0$ and there exist the properties (4·6a) and (4·6b), the following relations may be self-evident :

$$\hat{S}_i^j|m\rangle = 0 \ , \quad (j > i) \ , \qquad \hat{T}_q^p|m\rangle = 0 \ . \quad (q > p) \tag{4·8}$$

Certainly, we can learn that there exists a state, which satisfies the relations (4·1a) and (4·1b), in the present boson space.

Our next task is to show that the state $|m\rangle$ given in Eq.(4·7) is the eigenstate of the operators $\hat{S}_i^i$ and $\hat{T}_p^p$. For this aim, the following relations, which are proved in Appendix B, are useful :

$$[\hat{S}_i^i, \hat{B}_r^*] = \begin{cases} 0 \ , & (r = 1, 2, \cdots, M-i-1) \\ -\hat{B}_r^* \ , & (r = M-i, \cdots, M) \end{cases} \tag{4·9a}$$

$$[\hat{T}_p^p, \hat{B}_r^*] = \begin{cases} 0 \ , & (r = 1, 2, \cdots, p-1) \\ \hat{B}_r^* \ , & (r = p, p+1, \cdots, M) \end{cases} \tag{4·9b}$$

Then, it easily shown that the state $|m\rangle$ is the eigenstate of $\hat{S}_i^i$ and $\hat{T}_p^p$:

$$\hat{S}_i^i|m\rangle = -s_i|m\rangle \ , \qquad s_i = \sum_{k=1}^{i}\nu_k + \nu \ , \tag{4·10a}$$

$$\begin{aligned} \hat{T}_p^p|m\rangle = +t_p|m\rangle \ , \qquad t_p &= (M+1) + \sum_{k=1}^{M-p+1}\nu_k + \nu \\ &= (M+1) + s_{M-p+1} \ . \end{aligned} \tag{4·10b}$$

Since $s_i - s_{i-1} = \sum_{k=1}^{i}\nu_k - \sum_{k=1}^{i-1}\nu_k = \nu_i \ (i \geq 2)$ and $s_1 = \nu_1 + \nu$, we have

$$\begin{aligned} |m\rangle =& (\hat{B}_1^*)^{s_M - s_{M-1}}(\hat{B}_2^*)^{s_{M-1}-s_{M-2}}\cdots(\hat{B}_{M-k+1}^*)^{s_k - s_{k-1}} \\ & \cdots(\hat{B}_{M-1}^*)^{s_2 - s_1}(\hat{B}_M^*)^{s_1-\nu}(\hat{b}^*)^{\nu}|0\rangle \ . \end{aligned} \tag{4·11}$$

Further, the state (4·11) can be rewritten in another form :

$$|m\rangle = (\hat{A}_M^*)^{s_1-\nu}|s_1, s_2, \cdots, s_M\rangle \ , \tag{4·12a}$$

$$\begin{aligned} |s_1, s_2, \cdots, s_M\rangle = & (\hat{B}_1^*)^{s_M - s_{M-1}}(\hat{B}_2^*)^{s_{M-1}-s_{M-2}} \\ & \cdots(\hat{B}_{M-k+1}^*)^{s_k - s_{k-1}}\cdots(\hat{B}_{M-1}^*)^{s_2-s_1}(\hat{b}^*)^{s_1}|0\rangle \ . \end{aligned} \tag{4·12b}$$

Here, the operator $\hat{A}_M^*$ is defined in the following form :

$$\hat{A}_M^* = \hat{B}_M^* \hat{b} + \sum_{l=1}^{M} \hat{B}_M^*(l) \hat{a}_l \ . \tag{4·13}$$

Since $\hat{a}_l | s_1, s_2, \cdots, s_M\rangle = 0$, the state (4·12) returns to the state (4·11) for any form of $\hat{B}_M^*(l)$. However, we choose $\hat{B}_M^*(l)$ in the form that $\hat{A}_M^*$ commutes with all the generators :

$$[\hat{S}^i, \hat{A}_M^*] = [\hat{S}_i, \hat{A}_M^*] = [\hat{S}_i^j, \hat{A}_M^*] = [\hat{T}^p, \hat{A}_M^*] = [\hat{T}_p, \hat{A}_M^*] = [\hat{T}_p^q, \hat{A}_M^*] = 0 \ . \tag{4·14}$$

The operator $\hat{B}_M^*(l)$ which satisfies the condition (4·14) is given in the form that all the elements of the $(M-l+1)$-th row in $\hat{B}_r^*$ given in Eq.(4·4) for $r = M$, $\hat{b}_l^{r*}$ $(r = 1, 2, \cdots, M)$, are replaced with $\hat{a}^r$:

$$\hat{B}_M^*(l) = \begin{vmatrix} \hat{b}_M^{1*} & \hat{b}_M^{2*} & \cdots & \hat{b}_M^{M*} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{b}_{M-l+2}^{1*} & \hat{b}_{M-l+2}^{2*} & \cdots & \hat{b}_{M-l+2}^{M*} \\ \hat{a}^1 & \hat{a}^2 & \cdots & \hat{a}^M \\ \hat{b}_{M-l}^{1*} & \hat{b}_{M-l}^{2*} & \cdots & \hat{b}_{M-l}^{M*} \\ \cdots & \cdots & \cdots & \cdots \\ \hat{b}_1^{1*} & \hat{b}_1^{2*} & \cdots & \hat{b}_1^{M*} \end{vmatrix} \ . \quad (l = 1, 2, \cdots, M) \tag{4·15}$$

The proof is sketched in Appendix B. The property (4·14) will play an essential role in §7. All ν_i in the state (4·7) should be positive and, then, the state (4·11) and its reform (4·12) give us the following relations :

$$0 \leq s_1 \leq s_2 \leq \cdots \leq s_M \ , \tag{4·16}$$

$$0 \leq \nu \leq s_1 \ . \tag{4·17}$$

Thus, we could prove the existence of the state $|m\rangle$ which obeys the conditions (4·1a) and (4·1b) with the explicit form. In the same meaning as that mentioned in §3, $|m\rangle$ can be regarded as the intrinsic state. The state $|m\rangle$ is the eigenstate of $\hat{S}$ introduced in Eq.(2·3) and its eigenvalue S is given as

$$S = \sum_{i=1}^{M} (\nu - s_i) + \nu \ . \tag{4·18a}$$

Therefore, we have

$$\nu = (M+1)^{-1} \left(S + \sum_{i=1}^{M} s_i \right) \ , \quad \text{i.e.,} \ \ s_1 - \nu = s_1 - (M+1)^{-1} \left(S + \sum_{i=1}^{M} s_i \right) \ . \tag{4·18b}$$

Then, $|m\rangle$ is specified as

$$|m\rangle = |S, s_1, s_2, \cdots, s_M\rangle \ . \tag{4·19}$$

The relation (4·17) gives us the relation

$$-\sum_{i=1}^{M} s_i \leq S \leq (M+1)s_1 - \sum_{i=1}^{M} s_i \ . \tag{4·20}$$

Finally, we will discuss the eigenvalues of the Casimir operators $\hat{\Gamma}_{su(M+1)}$ and $\hat{\Gamma}_{su(M,1)}$. The eigenvalues $\gamma_{su(M+1)}^{(b)}$ and $\gamma_{su(M,1)}^{(b)}$ for the state $|m\rangle$ are given as

$$\gamma_{su(M+1)}^{(b)} = \sum_{i=1}^{M} s_i^2 - (M+1)^{-1}\left(\sum_{i=1}^{M} s_i\right)^2 - \sum_{i=1}^{M}(M-2i)s_i \ , \tag{4·21a}$$

$$\gamma_{su(M,1)}^{(b)} = \sum_{p=1}^{M} t_p^2 - (M+1)^{-1}\left(\sum_{p=1}^{M} t_p\right)^2 - \sum_{p=1}^{M}(M-2p)t_p \ . \tag{4·21b}$$

Of course, in the present, the case $N = M$ is treated. In the same way as that mentioned in §3, both are expressed as

$$\gamma_{su(M+1)}^{(b)} = \sum_{k=1}^{M-1}[k/(k+1)]s^k[s^k + (k+1)] + [M/(M+1)]s^M[s^M + (M+1)] \ , \tag{4·22a}$$

$$\gamma_{su(M,1)}^{(b)} = \sum_{r=1}^{M-1}[r/(r+1)]t^r[t^r + (r+1)] + [M/(M+1)]t^M[t^M - (M+1)] \ , \tag{4·22b}$$

$$s^k = \begin{cases} -(1/k)\sum_{i=1}^{k} s_i + s_{k+1} \ , & (k = 1, 2, \cdots, M-1) \\ (1/M)\sum_{i=1}^{M} s_i \ , & (k = M) \end{cases} \tag{4·23a}$$

$$t^r = \begin{cases} (1/r)\sum_{p=1}^{r} t_p - t_{r+1} \ , & (r = 1, 2, \cdots, M-1) \\ (1/M)\sum_{p=1}^{M} t_p \ , & (r = M) \end{cases} \tag{4·23b}$$

The negative sign in the form (4·23b) characterizes the $su(M,1)$ algebra. As is clear from the relation (4·10b), t^r can be expressed in terms of s^i. The state $|m\rangle$ is also specified as

$$|m\rangle = |S, s^1, s^2, \cdots, s^M\rangle \ . \tag{4·24}$$

§5. The generalized Lipkin model in the Schwinger boson representation

In §3, we formulate the generalized Lipkin model in the fermion space and showed various properties of the intrinsic state $|m\rangle\rangle$ $(= |(\gamma); n, n_1, \cdots, n_M\rangle\rangle)$. Further, in §4, the intrinsic state for $su(M+1)$ algebraic model (the $su(M,1)$ algebraic model), $|m\rangle$ $(= |S, s_1, \cdots, s_M\rangle)$ was investigated. The aim of this section is to reinvestigate the intrinsic state $|m\rangle$ in terms of $(n, n_1, \cdots, n_M)$ or $(N, n_1, \cdots, n_M)$.

First, we note two operators $\hat{S}$ and $\widetilde{N}$. Both are commuted with any generator of the $su(M+1)$ algebras expressed in terms of boson and fermion operators and $\hat{S}$ is commuted with any generator of the $su(M,1)$ algebra. However, there exists an essential difference between $\hat{S}$ and $\widetilde{N}$: $\hat{S}$ is not positive-definite, but, $\widetilde{N}$ is positive-definite. Therefore, it may be impossible to regard $\hat{S}$ as the counterpart of $\widetilde{N}$ and, then, we treat total fermion number N as a parameter in the Schwinger boson representation. However, from the relations (3·6) and (4·10a), it may be permitted to set up

$$\sigma_i = s_i \ . \quad (i = 1, 2, \cdots, M) \tag{5·1}$$

Then, we introduce the quantities $(\varOmega - n)$ and n_i into the Schwinger boson representation through the relations

$$\varOmega - n = (M+1)^{-1}\left(N + \sum_{j=1}^{M} s_j\right) \ , \tag{5·2a}$$

$$n_i = (M+1)^{-1}\left(N + \sum_{j=1}^{M} s_j\right) - s_i \ . \tag{5·2b}$$

Inversely, we have

$$N = \varOmega - n + \sum_{i=1}^{M} n_i \ , \tag{5·3a}$$

$$s_i = \varOmega - n - n_i \ . \tag{5·3b}$$

With the use of the relations (5·2), the inequality (4·16) can be rewritten as

$$\varOmega - n \geq n_1 \geq n_2 \geq \cdots \geq n_M \ . \tag{5·4}$$

The above is nothing but the inequality (3·14). Further, the inequality (4·20) is rewritten in the form

$$N - (M+1)(\varOmega - n) \leq S \leq N - (M+1)n_1 \ . \tag{5·5}$$

The relations (4·18a) and (5·2) give us

$$s_1 - \nu = (N - S)/(M+1) - n_1 \ , \tag{5·6}$$

$$s_k - s_{k-1} = n_k - n_{k-1} \ . \quad (k = 2, 3, \cdots, M) \tag{5·7}$$

Then, the state $|m\rangle$ shown in the relation (4·12) can be re-expressed in the form

$$|m\rangle = (\hat{A}_M^*)^{(N-S)/(M+1)-n_1}||s_1, s_2, \cdots, s_M\rangle \ , \tag{5·8a}$$

$$\begin{aligned} ||s_1, s_2, \cdots, s_M\rangle = (\hat{B}_1^*)^{n_{M-1}-n_M}(\hat{B}_2^*)^{n_{M-1}-n_{M-1}} \\ \cdots(\hat{B}_{M-k+1}^*)^{n_{k-1}-n_k}\cdots(\hat{B}_{M-1}^*)^{n_1-n_2}(\hat{b}^*)^{\Omega-n-n_1}|0\rangle \ . \end{aligned} \tag{5·8b}$$

The above shows that, with the use of the quantum numbers specifying the intrinsic state of the generalized Lipkin model for the fermion system, the explicit form is given in the Schwinger boson representation. It should be noted that, for a given set $(n, n_1, n_2, \cdots, n_M)$, there exist many states which are orthogonal with one another for different values of S obeying the inequality (5·5). The quantities s^k and t^r given in the relations (4·23a) and (4·23b), respectively, are expressed as

$$s^k = \begin{cases} (1/k)\displaystyle\sum_{i=1}^{k} n_i - n_{k+1} \ , \quad (k = 1, 2, \cdots, M-1) \\ \Omega - n - (1/M)\displaystyle\sum_{i=1}^{M} n_i \ , \quad (k = M) \end{cases} \tag{5·9a}$$

$$t^r = \begin{cases} n_{M-r} - (1/r)\displaystyle\sum_{i=M-r+1}^{M} n_i \ , \quad (r = 1, 2, \cdots, M-1) \\ M + 1 + \Omega - n - (1/M)\displaystyle\sum_{i=1}^{M} n_i \ , \quad (r = M) \end{cases} \tag{5·9b}$$

The inequality (5·4) tells us that all s^k and t^r are positive.

§6. General scheme for constructing orthogonal set for the irreducible representation

Until the previous section, we have investigated the method how to construct the intrinsic state $|m\rangle$. Then, our next task is to find the orthogonal set built on the state $|m\rangle$ by operating $\hat{S}^i$, $\hat{S}_j^i$, $\hat{T}^p$ and $\hat{T}_p^q$ as the raising operators. Here, $i = 1, 2, \cdots, M$, $p = 1, 2, \cdots, M$, $j > i$ and $q > p$. The problem is to determine the ordering the operation of $\hat{S}^i$, $\hat{S}_j^i$, $\hat{T}^p$ and $\hat{T}_p^q$. Of course, there does not exist any problem between $(\hat{S}^i, \hat{S}_j^i)$ and $(\hat{T}^p, \hat{T}_p^q)$, because these commute with each other. In the sense of the full use of the raising operators, the present idea is similar to a method developed by Moshinsky for the shell model.[10)]

First, we investigate the operation of $\hat{S}^i$ $(i = 1, 2, \cdots, M)$. For this aim, let us define a set of the operators $\hat{\boldsymbol{S}}^i$ expressed in terms of the linear combinations for $\hat{S}^l$:

$$\hat{\boldsymbol{S}}^i = \sum_{l=1}^{M} \hat{S}^l \hat{U}_{il} \ . \quad (i = 1, 2, \cdots, M) \tag{6·1}$$

Here, $\hat{U}_{il}$ is a function of $\hat{S}_j^i$ $(j > i = 1, 2, \cdots, M-1)$ and $(\hat{S}_i^i - \hat{S}_1^1)$ $(i = 2, 3, \cdots, M)$. Under an appropriate choice of the coefficients $\hat{U}_{il}$, let $\hat{\boldsymbol{S}}^i$ satisfy the following relations :

$$[\hat{S}_i^j, \hat{\boldsymbol{S}}^k] = \sum_{m>l=1,\cdots,M-1} \hat{V}_{ijk,lm} \hat{S}_l^m \ , \quad (j > i = 1, 2, \cdots, M-1) \tag{6·2a}$$

$$[\hat{S}_i^i, \hat{\boldsymbol{S}}^k] = (1 + \delta_{ik}) \hat{\boldsymbol{S}}^k \ , \quad (i = 1, 2, \cdots, M) \tag{6·2b}$$

$$[\hat{\boldsymbol{S}}^i, \hat{\boldsymbol{S}}^j] = 0 \ . \tag{6·2c}$$

Here, generally, $\hat{V}_{ijk,lm}$ is operators. For example, $\hat{\boldsymbol{S}}^1$, $\hat{\boldsymbol{S}}^2$ and $\hat{\boldsymbol{S}}^3$ are given in the form

$$\hat{\boldsymbol{S}}^1 = \hat{S}^1 \ , \tag{6·3}$$

$$\hat{\boldsymbol{S}}^2 = \hat{S}^1 \cdot \hat{S}_2^1 + \hat{S}^2 \cdot (\hat{S}_2^2 - \hat{S}_1^1) \ , \tag{6·4}$$

$$\begin{aligned} \hat{\boldsymbol{S}}^3 = {} & \hat{S}^1 \cdot \left[\hat{S}_2^1 \hat{S}_3^2 + \hat{S}_3^1 ((\hat{S}_3^3 - \hat{S}_1^1) - (\hat{S}_2^2 - \hat{S}_1^1)) \right] + \hat{S}^2 \cdot \hat{S}_3^2 \left[(\hat{S}_3^3 - \hat{S}_1^1) - 1 \right] \\ & + \hat{S}^3 \cdot \left[(\hat{S}_3^3 - \hat{S}_1^1) - (\hat{S}_2^2 - \hat{S}_1^1) \right] \left[(\hat{S}_3^3 - \hat{S}_1^1) - 1 \right] \ . \end{aligned} \tag{6·5}$$

With the use of the operator $\hat{\boldsymbol{S}}^i$, we define the following state :

$$|m(1)\rangle = \hat{\boldsymbol{C}}_1 (m_1^{(1)}, \cdots, m_M^{(1)}) |m\rangle \ , \tag{6·6a}$$

$$\hat{\boldsymbol{C}}_1 (m_1^{(1)}, \cdots, m_M^{(1)}) = (\hat{\boldsymbol{S}}^1)^{m_1^{(1)}} \cdots (\hat{\boldsymbol{S}}^M)^{m_M^{(1)}} \ . \tag{6·6b}$$

The relation (6·2b) shows us that the state $|m(1)\rangle$ is an eigenstate of $\hat{S}_i^i$:

$$\hat{S}_i^i |m(1)\rangle = - \left(s_i - m_i^{(1)} - \sum_{k=1}^{M} m_k^{(1)} \right) |m(1)\rangle \ . \quad (i = 1, 2, \cdots, M) \tag{6·7}$$

Therefore, the set $(|m(1)\rangle)$ forms an orthogonal set and the relation (6·2c) tells us that the ordering of $\hat{\boldsymbol{S}}^i$ in $\hat{\boldsymbol{C}}_1$ is arbitrary. Further, we note the following relations :

$$\hat{S}_i^j |m(1)\rangle = 0 \ , \qquad (j > i = 1, 2, \cdots, M-1) \tag{6·8a}$$

$$(\hat{S}_i^i - \hat{S}_1^1) |m(1)\rangle = -[(s_i - m_i^{(1)}) - (s_1 - m_1^{(1)})] |m(1)\rangle \ . \ \ (i = 2, 3, \cdots, M) \tag{6·8b}$$

The set $(\hat{S}_j^i - \delta_{ij} \hat{S}_1^1)$ $(i, j = 1, 2, \cdots, M)$ forms the $su(M)$ algebra as a sub-algebra of the starting $su(M+1)$ algebra. The relations (6·8a) and (6·8b) show us that the state $|m(1)\rangle$ is the intrinsic state of the $su(M)$ algebra and $\hat{S}_j^i$ $(j > i = 1, 2, \cdots, M-1)$ is the raising operator on the intrinsic state $|m(1)\rangle$. From the above consideration, it may be concluded that the operation of $\hat{S}^i$ on the state $|m\rangle$ was finished in the form (6·6).

Next task is how to operate $\hat{S}_j^i$ $(j > i = 1, 2, \cdots, M-1)$ on the state $|m(1)\rangle$. We already mentioned that the set $(\hat{S}_j^i - \delta_{ij} \hat{S}_1^1)$ $(i, j = 1, 2, \cdots, M)$ forms the $su(M)$ algebra and if its generators are decomposed to $(\hat{S}_i^1, \hat{S}_1^i, (i = 2, 3, \cdots, M), \hat{S}_2^2 - \hat{S}_1^1, \hat{S}_j^i - \delta_{ij} \hat{S}_2^2, (i, j =$

$2,3,\cdots M)$), the set $(\hat{S}^i_j-\delta_{ij}\hat{S}^2_2)$ $(i,j=2,3,\cdots,M)$ forms the $su(M-1)$ algebra. Under the above note, we consider the operation of $\hat{S}^1_i$ $(i=2,3,\cdots,M)$ on the state $|m(1)\rangle$. In the same form as that shown in Eq.(6·1), we introduce the operator $\hat{\boldsymbol{S}}^1_i$ in the form

$$\hat{\boldsymbol{S}}^1_i=\sum_{l=2}^{M}\hat{S}^1_l\hat{U}^{(1)}_{il}\ . \quad (i=2,3,\cdots,M) \tag{6·9}$$

Here, $\hat{U}^{(1)}_{il}$ is a function of $\hat{S}^i_j$ $(j>i=2,3,\cdots,M-1)$ and $(\hat{S}^i_i-\hat{S}^2_2)$ $(i=3,4,\cdots,M)$. In parallel to the relations (6·2), $\hat{\boldsymbol{S}}^1_i$ is regarded as the operator satisfying the relations

$$[\hat{S}^j_i,\hat{\boldsymbol{S}}^1_k]=\sum_{m>l=2,\cdots,M-1}\hat{V}^{(1)}_{ijk,lm}\hat{S}^m_l\ , \quad (j>i=2,3,\cdots,M-1) \tag{6·10a}$$

$$[\hat{S}^i_i-\hat{S}^1_1,\hat{\boldsymbol{S}}^1_k]=(1+\delta_{ik})\hat{\boldsymbol{S}}^1_k\ , \quad (i=2,3,\cdots,M) \tag{6·10b}$$

$$[\hat{\boldsymbol{S}}^1_i,\hat{\boldsymbol{S}}^1_j]=0\ . \tag{6·10c}$$

For example, $\hat{\boldsymbol{S}}^1_2$ and $\hat{\boldsymbol{S}}^1_3$ are given as

$$\hat{\boldsymbol{S}}^1_2=\hat{S}^1_2\ , \tag{6·11}$$

$$\hat{\boldsymbol{S}}^1_3=\hat{S}^1_2\cdot\hat{S}^2_3+\hat{S}^1_3\cdot(\hat{S}^3_3-\hat{S}^2_2)\ . \tag{6·12}$$

With the use of the operator $\hat{\boldsymbol{S}}^1_i$, we define the following state :

$$|m(2)\rangle=\hat{\boldsymbol{C}}_2(m^{(2)}_2,\cdots,m^{(2)}_M)|m(1)\rangle\ , \tag{6·13a}$$

$$\hat{\boldsymbol{C}}_2(m^{(2)}_2,\cdots,m^{(2)}_M)=(\hat{\boldsymbol{S}}^1_2)^{m^{(2)}_2}\cdots(\hat{\boldsymbol{S}}^1_M)^{m^{(2)}_M}\ . \tag{6·13b}$$

The state $|m(2)\rangle$ is an eigenstate of $(\hat{S}^i_i-\hat{S}^1_1)$:

$$(\hat{S}^i_i-\hat{S}^1_1)|m(2)\rangle=-\left[(s_i-m^{(1)}_i-m^{(2)}_i)-(s_1-m^{(1)}_1)-\sum_{k=2}^{M}m^{(2)}_k\right]|m(2)\rangle\ . \tag{6·14}$$

The above means that the set $(|m(2)\rangle)$ forms the orthogonal set. Thus, we finished the operation of $\hat{S}^1_i$ $(i=2,\cdots,M)$ through $\hat{\boldsymbol{S}}^1_i$.

Next operation is related with the operator $\hat{S}^2_i$ $(i=3,\cdots,M)$. In this case, we also note that the state $|m(2)\rangle$ is the intrinsic state for the $su(M-1)$ algebra. Therefore, we can apply the same idea as that already presented in the case of $\hat{S}^1_i$. By applying the above idea, successively, we arrive at the stage of the operation of $\hat{S}^n_i$ $(i=n+1,\cdots,M)$. First, the set $(\hat{S}^n_i,\hat{S}^i_n,\ (i=n+1,\cdots,M),\hat{S}^{n+1}_{n+1}-\hat{S}^n_n,\hat{S}^i_j-\delta_{ij}\hat{S}^{n+1}_{n+1},\ (i,j=n+1,\cdots,M))$ forms the $su(M-n+1)$ algebra and the sub-set $(\hat{S}^i_j-\delta_{ij}\hat{S}^{n+1}_{n+1})$ $(i,j=n+1,\cdots,M)$ composes the $su(M-n)$ algebra. In this case, we also define the operator $\hat{\boldsymbol{S}}^n_i$ in the form

$$\hat{\boldsymbol{S}}^n_i=\sum_{l=n+1}^{M}\hat{S}^n_l\hat{U}^{(n)}_{il}\ . \quad (i=n+1,\cdots,M) \tag{6·15}$$

Here, $\hat{U}_{il}^{(n)}$ is a function of $\hat{S}_j^i$ $(j > i = n+1, \cdots, M-1)$ and $(\hat{S}_i^i - \hat{S}_{n+1}^{n+1})$ $(i = n+2, \cdots, M)$. Of course, we regard $\hat{\boldsymbol{S}}_i^n$ as the operator satisfying the relations

$$[\hat{S}_i^j, \hat{\boldsymbol{S}}_k^n] = \sum_{m>l=n+1,\cdots,M-1} \hat{V}_{ijk,lm}^{(n)} \hat{S}_l^m \ , \quad (j > i = n+1, \cdots, M-1) \tag{6·16a}$$

$$[\hat{S}_i^i - \hat{S}_n^n, \hat{\boldsymbol{S}}_k^n] = (1 + \delta_{ik}) \hat{\boldsymbol{S}}_k^n \ , \quad (i = n+1, \cdots, M) \tag{6·16b}$$

$$[\hat{\boldsymbol{S}}_i^n, \hat{\boldsymbol{S}}_j^n] = 0 \ . \tag{6·16c}$$

The relations (6·15) and (6·16) in the case $n = 1$ are reduced to Eqs.(6·9) and (6·10). In the same way as that in the case $n = 1$, the state $|m(n+1)\rangle$ is defined as

$$|m(n+1)\rangle = \hat{\boldsymbol{C}}_{n+1}(m_{n+1}^{(n+1)}, \cdots, m_M^{(n+1)}) |m(n)\rangle \ , \tag{6·17a}$$

$$\hat{\boldsymbol{C}}_{n+1}(m_{n+1}^{(n+1)}, \cdots, m_M^{(n+1)}) = (\hat{\boldsymbol{S}}_{n+1}^n)^{m_{n+1}^{(n+1)}} \cdots (\hat{\boldsymbol{S}}_M^n)^{m_M^{(n+1)}} \ . \tag{6·17b}$$

Under the successive application of the above idea, finally, we arrive at the case $n = M-1$. In this case, the set $(\hat{S}_M^{M-1}, \hat{S}_{M-1}^M, \hat{S}_M^M - \hat{S}_{M-1}^{M-1})$ forms the $su(2)$ algebra and $\hat{\boldsymbol{S}}_i^n$ is reduced to $\hat{\boldsymbol{S}}_M^{M-1} = \hat{S}_M^{M-1}$.

By summarizing the above procedure, the state $|m(M)\rangle$ is expressed in the form

$$\begin{aligned} |m(M)\rangle = {} & \hat{\boldsymbol{C}}_M(m_M^{(M)}) \hat{\boldsymbol{C}}_{M-1}(m_{M-1}^{(M-1)}, m_M^{(M-1)}) \\ & \times \cdots \times \hat{\boldsymbol{C}}_{n+1}(m_{n+1}^{(n+1)}, \cdots, m_M^{(n+1)}) \\ & \times \cdots \times \hat{\boldsymbol{C}}_2(m_2^{(2)}, \cdots, m_M^{(2)}) \hat{\boldsymbol{C}}_1(m_1^{(1)}, \cdots, m_M^{(1)}) |m\rangle \ . \end{aligned} \tag{6·18}$$

The above method can be applied to the case of the $su(M,1)$ algebra by replacing $\hat{S}^i$, $\hat{S}_i$, $\hat{S}_j^i$ $(j > i)$ and $\hat{S}_i^j$ $(j > i)$ with $\hat{T}^p$, $\hat{T}_p$, $\hat{T}_p^q$ $(q > p)$ and $\hat{T}_q^p$ $(q > p)$, respectively. The state $|m\mu(M)\rangle$ in the $su(M+1)$ and in the $su(M,1)$ algebra can be expressed in the form

$$\begin{aligned} |m\mu(M)\rangle = {} & \hat{\boldsymbol{D}}_M(\mu_M^{(M)}) \cdots \hat{\boldsymbol{D}}_1(\mu_1^{(1)}, \cdots, \mu_M^{(1)}) \\ & \times \hat{\boldsymbol{C}}_M(m_M^{(M)}) \cdots \hat{\boldsymbol{C}}_1(m_1^{(1)}, \cdots, m_M^{(1)}) |n, n_1, \cdots, n_M\rangle \ . \end{aligned} \tag{6·19}$$

It may be not necessary to mention the meaning of $\hat{\boldsymbol{D}}_M(\mu_M^{(M)}) \cdots \hat{\boldsymbol{D}}_1(\mu_1^{(1)}, \cdots, \mu_M^{(1)})$. Clearly, the state $|m\mu(M)\rangle$ contains $(M+1)^2$ quantum numbers and our problem is reduced to determine the operators $\hat{U}_{il}$ and $\hat{U}_{il}^{(n)}$ $(n = 1, 2, \cdots, M-1)$ appearing in Eqs.(6·1) and (6·15), respectively. Some examples were already given in Eqs.(6·3), (6·4), (6·5), (6·11) and (6·12). Further, we should note that, in the form of $|m\rangle$ shown in Eq.(5·8), $\hat{A}_M^*$ obeys the relation (4·14). Therefore, the part $|n, n_1, \cdots, n_M\rangle$ shown in Eq.(6·19) is replaced with

$$|n, n_1, \cdots, n_M\rangle = (\hat{A}_M^*)^{(N-S)/(M+1)-n_1} ||s_1, s_2, \cdots, s_M\rangle \ . \tag{6·20}$$

Since we have the relation (4·14), $|m\mu(M)\rangle$ can be rewritten in the form

$$
\begin{aligned}
|m\mu(M)\rangle &= (\hat{A}_M^*)^{(N-S)/(M+1)-n_1}||m\mu(M)\rangle \ , \\
||m\mu(M)\rangle &= \hat{\boldsymbol{D}}_M(\mu_M^{(M)})\cdots\hat{\boldsymbol{D}}_1(\mu_1^{(1)},\cdots,\mu_M^{(1)}) \\
&\qquad \times\hat{\boldsymbol{C}}_M(m_M^{(M)})\cdots\hat{\boldsymbol{C}}_1(m_1^{(1)},\cdots,m_M^{(1)})||s_1,s_2,\cdots,s_M\rangle \ .
\end{aligned}
\tag{6·21}
$$

The role of the part related with $\hat{A}_M^*$ in the state $|m\mu(M)\rangle$ can be interpreted in a way mentioned below. We note the relation

$$
[\hat{A}_M^*, \hat{S}] = (M+1)\hat{A}_M^* \ . \tag{6·22}
$$

As was shown in the inequality (5·5), S runs in the region between $S_{\max}$ and $S_{\min}$:

$$
S_{\max} = N-(M+1)n_1 \ , \qquad S_{\min} = N-(M+1)(\varOmega-n) \ . \tag{6·23}
$$

If $S=S_{\max}$, $(N-S)/(M+1)-n_1=0$ and $|m\mu(M)\rangle = ||m\mu(M)\rangle$. By ρ-time operation of $\hat{A}_M^*$ on $||m\mu(M)\rangle$, we have the state with $S=S_{\max}-(M+1)\rho$ and, finally, the s_1-time operation gives us the state with $S=S_{\min}$. Therefore, $\hat{A}_M^*$ plays a role of the lowering operator for $\hat{S}$. For obtaining the representation of the $su(M+1)$ and the $su(M,1)$ algebra, it is enough to take into account the state $||m\mu(M)\rangle$.

§7. Discussion and concluding remarks

In this section, mainly, we will discuss two concrete examples, the $su(2)$ and the $su(3)$ algebra. Let us start from the case of the $su(2)$ algebra. This case corresponds to $M=N=1$ and the $su(2)$ and the $su(1,1)$ generators are expressed in the following form :

$$
\begin{aligned}
&\hat{S}^1 = \hat{a}_1^*\hat{b} + \hat{a}^{1*}\hat{b}_1^1 \ , \qquad \hat{S}^2 = \hat{b}^*\hat{a}_1 + \hat{b}_1^{1*}\hat{a}^1 \ , \\
&\hat{S}_1^1 = \hat{a}_1^*\hat{a}_1 - \hat{b}_1^{1*}\hat{b}_1^1 + \hat{a}^{1*}\hat{a}^1 - \hat{b}^*\hat{b} \ ,
\end{aligned}
\tag{7·1a}
$$

$$
\begin{aligned}
&\hat{T}^1 = \hat{a}_1^*\hat{b}^* - \hat{a}_1^*\hat{b}_1^{1*} \ , \qquad \hat{T}^2 = \hat{b}\hat{a}_1 - \hat{b}_1^1\hat{a}_1 \ , \\
&\hat{T}_1^1 = \hat{a}^{1*}\hat{a}^1 + \hat{b}_1^{1*}\hat{b}_1^1 + \hat{a}_1^*\hat{a}_1 + \hat{b}^*\hat{b} + 2 \ .
\end{aligned}
\tag{7·1b}
$$

The above algebras are constructed in terms of four kinds of the boson operators $(\hat{a}_1, \hat{a}_1^*)$, $(\hat{b}, \hat{b}^*)$, $(\hat{a}^1, \hat{a}^{1*})$ and $(\hat{b}_1^1, \hat{b}_1^{1*})$ and the form (7·1a) should be compared with the form (1·13). The operator $\hat{S}$ is expressed as

$$
\hat{S} = \hat{a}_1^*\hat{a}_1 - \hat{a}^{1*}\hat{a}^1 - \hat{b}_1^{1*}\hat{b}_1^1 + \hat{b}^*\hat{b} \ . \tag{7·2}
$$

Further, $\hat{A}_1^*$ is given in the form

$$
\hat{A}_1^* = \hat{B}_1^*\hat{b} + \hat{B}_1^*(1)\hat{a}_1 = \hat{b}_1^{1*}\hat{b} + \hat{a}^{1*}\hat{a}_1 \ . \tag{7·3}
$$

Since we treat the case $M=1$, $\hat{\boldsymbol{D}}_1(\mu_1^{(1)})=(\hat{T}^1)^{\mu_1^{(1)}}$ and $\hat{\boldsymbol{C}}_1(m_1^{(1)})=(\hat{S}^1)^{m_1^{(1)}}$. Therefore, $|m\mu(1)\rangle$ can be expressed in the following form :

$$|m\mu(1)\rangle=(\hat{A}_1^*)^{(s_1-S)/2}(\hat{T}^1)^{(\mu-(s_1+2))/2}(\hat{S}^1)^{(s_1+m)/2}(\hat{b}^*)^{s_1}|0\rangle\ . \tag{7·4}$$

Here,

$$\begin{aligned}
&(N-S)/2-n_1=(s_1-S)/2\ ,\\
&\mu_1^{(1)}=(\mu-(s_1+2))/2\ ,\qquad m_1^{(1)}=(s_1+m)/2\ ,\\
&s_1=\Omega-n-n_1\ .
\end{aligned}\tag{7·5}$$

The state $|m\mu(1)\rangle$ is the eigenstate of $\hat{S}$, $\hat{\Gamma}_{su(1,1)}$, $\hat{T}_1^1$, $\hat{\Gamma}_{su(2)}$ and $\hat{S}_1^1$ with the eigenvalues S, $(1/2)(s_1+2)((s_1+2)-2)$, μ, $(1/2)s_1(s_1+2)$ and m, respectively. Of course, $\mu=s_1+2, s_1+4, s_1+6,\cdots$ and $m=-s_1,-s_1+2,\cdots,s_1-2,s_1$. Since $S_{\min}=N-2(\Omega-n)=-s_1$ and $S_{\max}=N-2n_1=s_1$, we have

$$S=-s_1,-s_1+2,\cdots,s_1-2,s_1\ . \tag{7·6}$$

Concerning with the relation (7·6), we must give an interesting remark. The following set obeys the $su(2)$ algebra :

$$\hat{R}^1=\hat{A}_1^*\ ,\qquad \hat{R}_1=\hat{A}_1\ ,\qquad \hat{R}_1^1=-\hat{S}\ . \tag{7·7}$$

Of course, the set $(\hat{R}^1,\hat{R}_1,\hat{R}_1^1)$ commutes with $(\hat{S}^1,\hat{S}_1,\hat{S}_1^1)$ and $(\hat{T}^1,\hat{T}_1,\hat{T}_1^1)$ and the state $|m\mu(1)\rangle$ is the eigenstate of the Casimir operator $\hat{\Gamma}'_{su(2)}$ $(=\hat{R}^1\hat{R}_1+\hat{R}_1\hat{R}^1+(1/2)(\hat{R}_1^1)^2)$ with the eigenvalue $(1/2)s_1(s_1+2)$. The details of the above three algebras have been already discussed by three of the present authors (A. K., J. P. & M. Y.).[11]

Our next interest is concerned with the case $M=N=2$. The basic operators can be expressed in the following form :

$$\begin{aligned}
&\hat{S}^1=\hat{a}_1^*\hat{b}+\hat{a}^{1*}\hat{b}_1^1+\hat{a}^{2*}\hat{b}_1^2\ ,\qquad \hat{S}_1=\hat{b}^*\hat{a}_1+\hat{b}_1^{1*}\hat{a}^1+\hat{b}_1^{2*}\hat{a}^2\ ,\\
&\hat{S}^2=\hat{a}_2^*\hat{b}+\hat{a}^{1*}\hat{b}_2^1+\hat{a}^{2*}\hat{b}_2^2\ ,\qquad \hat{S}_2=\hat{b}^*\hat{a}_2+\hat{b}_2^{1*}\hat{a}^1+\hat{b}_2^{2*}\hat{a}^2\ ,\\
&\hat{S}_2^1=\hat{a}_2^*\hat{a}_1-\hat{b}_1^{1*}\hat{b}_2^1-\hat{b}_1^{2*}\hat{b}_2^2\ ,\qquad \hat{S}_1^2=\hat{a}_1^*\hat{a}_2-\hat{b}_2^{1*}\hat{b}_1^1-\hat{b}_2^{2*}\hat{b}_1^2\ ,\\
&\hat{S}_1^1=\hat{a}_1^*\hat{a}_1-\hat{b}_1^{1*}\hat{b}_1^1-\hat{b}_1^{2*}\hat{b}_1^2+\hat{a}^{1*}\hat{a}^1+\hat{a}^{2*}\hat{a}^2-\hat{b}^*\hat{b}\ ,\\
&\hat{S}_2^2=\hat{a}_2^*\hat{a}_2-\hat{b}_2^{1*}\hat{b}_2^1-\hat{b}_2^{2*}\hat{b}_2^2+\hat{a}^{1*}\hat{a}^1+\hat{a}^{2*}\hat{a}^2-\hat{b}^*\hat{b}\ ,
\end{aligned}\tag{7·8a}$$

$$\begin{aligned}
&\hat{T}^1=\hat{a}^{1*}\hat{b}^*-\hat{a}_1^*\hat{b}_1^{1*}-\hat{a}_2^*\hat{b}_2^{1*}\ ,\qquad \hat{T}_1=\hat{b}\hat{a}^1-\hat{b}_1^1\hat{a}_1-\hat{b}_2^1\hat{a}_2\ ,\\
&\hat{T}^2=\hat{a}^{2*}\hat{b}^*-\hat{a}_1^*\hat{b}_1^{2*}-\hat{a}_2^*\hat{b}_2^{2*}\ ,\qquad \hat{T}_2=\hat{b}\hat{a}^2-\hat{b}_1^2\hat{a}_1-\hat{b}_2^2\hat{a}_2\ ,\\
&\hat{T}_1^2=\hat{a}^{2*}\hat{a}^1+\hat{b}_1^{2*}\hat{b}_1^1+\hat{b}_2^{2*}\hat{b}_2^1\ ,\qquad \hat{T}_2^1=\hat{a}^{1*}\hat{a}^2+\hat{b}_1^{1*}\hat{b}_1^2+\hat{b}_2^{1*}\hat{b}_2^2\ ,\\
&\hat{T}_1^1=\hat{a}^{1*}\hat{a}^1+\hat{b}_1^{1*}\hat{b}_1^1+\hat{b}_2^{1*}\hat{b}_2^1+\hat{a}_1^*\hat{a}_1+\hat{a}_2^*\hat{a}_2+\hat{b}^*\hat{b}+3\ ,\\
&\hat{T}_2^2=\hat{a}^{2*}\hat{a}^2+\hat{b}_1^{2*}\hat{b}_1^2+\hat{b}_2^{2*}\hat{b}_2^2+\hat{a}_1^*\hat{a}_1+\hat{a}_2^*\hat{a}_2+\hat{b}^*\hat{b}+3\ ,
\end{aligned}\tag{7·8b}$$

$$\hat{S} = \hat{a}_1^*\hat{a}_1 + \hat{a}_2^*\hat{a}_2 - \hat{a}^{1*}\hat{a}^1 - \hat{a}^{2*}\hat{a}^2 - \hat{b}_1^{1*}\hat{b}_1^1 - \hat{b}_2^{1*}\hat{b}_2^1 - \hat{b}_1^{2*}\hat{b}_1^2 - \hat{b}_2^{2*}\hat{b}_2^2 \ , \tag{7·9}$$
$$\hat{A}_2^* = (\hat{b}_2^{1*}\hat{b}_1^{2*} - \hat{b}_1^{1*}\hat{b}_2^{2*})\hat{b} + (\hat{b}_1^{2*}\hat{a}^{1*} - \hat{b}_1^{1*}\hat{a}^{2*})\hat{a}_1 + (\hat{b}_2^{1*}\hat{a}^{2*} - \hat{b}_2^{2*}\hat{a}^{1*})\hat{a}_2 \ , \tag{7·10}$$
$$\hat{\boldsymbol{S}}^1 = \hat{S}^1 \ , \qquad \hat{\boldsymbol{S}}^2 = \hat{S}^1 \cdot \hat{S}_2^1 + \hat{S}^2 \cdot (\hat{S}_2^2 - \hat{S}_1^1) \ , \qquad \hat{\boldsymbol{S}}_2^1 = \hat{S}_2^1 \ , \tag{7·11}$$
$$\hat{\boldsymbol{T}}^1 = \hat{T}^1 \ , \qquad \hat{\boldsymbol{T}}^2 = \hat{T}^1 \cdot \hat{T}_1^2 + \hat{T}^2 \cdot (\hat{T}_2^2 - \hat{T}_1^1) \ , \qquad \hat{\boldsymbol{T}}_1^2 = \hat{T}_1^2 \ . \tag{7·12}$$

With the use of the above basic operators, let us construct the orthogonal set for the present case. For this aim, the quantum numbers shown in the relations (5·9) are convenient :

$$s^1 = n_1 - n_2 \ , \qquad s^2 = \Omega - n - (n_1 + n_2)/2 \ , \tag{7·13}$$
$$t^1 = s^1 \ , \qquad t^2 = s^2 + 3 \ . \tag{7·14}$$

Inversely, n_1 and n_2 are expressed as

$$n_1 = \Omega - n + s^1/2 - s^2 \ , \qquad n_2 = \Omega - n - s^1/2 - s^2 \ . \tag{7·15}$$

The eigenvalues of the $su(3)$ and the $su(2,1)$ algebra are given in the form

$$\begin{aligned} \gamma_{su(3)}^{(b)} &= (1/2)s^1(s^1+2) + (2/3)s^2(s^2+3) \ , \\ \gamma_{su(2,1)}^{(b)} &= (1/2)s^1(s^1+2) + (2/3)(s^2+3)[(s^2+3) - 3] \end{aligned} \tag{7·16}$$

Since $n_1 - n_2 = s^1$ and $\Omega - n - n_1 = s^2 - s^1/2$, $||s_1, s_2\rangle$ is expressed as

$$||s_1, s_2\rangle = (\hat{b}_2^{1*})^{s^1}(\hat{b}^*)^{(2s^2 - s^1)/2}|0\rangle \ . \tag{7·17}$$

Further, the quantity $(N-S)/3 - n_1$ is given as

$$(N-S)/3 - n_1 = (s^2 - S)/3 - s^1/2 \ , \tag{7·18}$$

$$S_{\min} = N - 3(\Omega - n) = -2s^2 \ , \tag{7·19a}$$
$$S_{\max} = N - 3n_1 = 3(s^2/3 - s^1/2) \ . \tag{7·19b}$$

Therefore, we have

$$0 \le (N-S)/3 - n_1 \le (2s^2 - s^1)/2 \ . \tag{7·20}$$

The operator operated on the intrinsic state can be expressed in the form

$$\begin{aligned} &\hat{\boldsymbol{D}}_2(\mu_2^{(2)})\hat{\boldsymbol{D}}_1(\mu_1^{(1)}, \mu_2^{(1)})\hat{\boldsymbol{C}}_2(m_2^{(2)})\hat{\boldsymbol{C}}_1(m_1^{(1)}, m_2^{(1)}) \\ &= (\hat{\boldsymbol{T}}_1^2)^{\mu_2^{(2)}}(\hat{\boldsymbol{T}}^1)^{\mu_1^{(1)}}(\hat{\boldsymbol{T}}^2)^{\mu_2^{(1)}}(\hat{\boldsymbol{S}}_2^1)^{m_2^{(2)}}(\hat{\boldsymbol{S}}^1)^{m_1^{(1)}}(\hat{\boldsymbol{S}}^2)^{m_2^{(1)}} \ . \end{aligned} \tag{7·21}$$

Therefore, the orthogonal set is obtained in the following form :

$$\begin{aligned}|m\mu(2)\rangle = &(\hat{A}_2^*)^{(s^2-S)/3-s^1/2}\\ &\times\hat{\boldsymbol{D}}_2(\mu_2^{(2)})\hat{\boldsymbol{D}}_1(\mu_1^{(1)},\mu_2^{(1)})\hat{\boldsymbol{C}}_2(m_2^{(2)})\hat{\boldsymbol{C}}_1(m_1^{(1)},m_2^{(1)})\\ &\times(\hat{b}_2^{1*})^{s^1}(\hat{b}^*)^{(2s^2-s^1)/2}|0\rangle\ . \end{aligned} \qquad (7{\cdot}22)$$

If $S=S_{\max}$ and $\mu_2^{(2)}=\mu_1^{(1)}=\mu_2^{(1)}=0$, the state (7·22) is expressed as

$$|m\mu(2)\rangle = (\hat{\boldsymbol{S}}_2^1)^{m_2^{(2)}}(\hat{\boldsymbol{S}}^1)^{m_1^{(1)}}(\hat{\boldsymbol{S}}^2)^{m_2^{(1)}}(\hat{b}_2^{1*})^{s^1}(\hat{b}^*)^{(2s^2-s^1)/2}|0\rangle\ . \qquad (7{\cdot}23)$$

The set (7·23) is the orthogonal set for the irreducible representation of the $su(2)$ algebra. Concerning the state (7·23), we will give a remark. The set $(\hat{S}_2^1, \hat{S}_1^2, \hat{S}_2^2-\hat{S}_1^1)$ forms the $su(2)$ algebra and we adopt the conventional notation for the $su(2)$ algebra :

$$\hat{J}_+ = \hat{S}_2^1\ , \qquad \hat{J}_- = \hat{S}_1^2\ , \qquad \hat{J}_0 = (\hat{S}_2^2-\hat{S}_1^1)/2\ . \qquad (7{\cdot}24)$$

Further, we can prove that $\hat{S}^1$ and $\hat{S}^2$ play a role of the spherical tensor with rank 1/2 and we denote them as

$$\hat{C}_{-1/2}^* = \hat{S}^1\ , \qquad \hat{C}_{+1/2}^* = \hat{S}^2\ . \qquad (7{\cdot}25)$$

Then, $\hat{\boldsymbol{S}}^1$ and $\hat{\boldsymbol{S}}^2$ are expressed as

$$\begin{aligned}&\hat{\boldsymbol{S}}^1 = \hat{C}_{-1/2}^* = \hat{\boldsymbol{C}}_{-1/2}^*\ ,\\ &\hat{\boldsymbol{S}}^2 = \hat{C}_{-1/2}^*\cdot\hat{J}_+ + 2\hat{C}_{+1/2}^*\cdot\hat{J}_0 = 2\hat{\boldsymbol{C}}_{+1/2}^*\ . \end{aligned}\qquad (7{\cdot}26)$$

With the use of the above notations, the state (7·23) is rewritten in the form (except the normalization)

$$|m\mu(2)\rangle = (\hat{J}_+)^{j_3+m_3}(\hat{\boldsymbol{C}}_{-1/2}^*)^{-j_3+j_2+j_1}(\hat{\boldsymbol{C}}_{+1/2}^*)^{j_3+j_2-j_1}||j_1, s^2\rangle\ . \qquad (7{\cdot}27\mathrm{a})$$

Here, we put

$$m_1^{(1)} = -j_3+j_2+j_1\ , \qquad m_2^{(1)} = j_3+j_2-j_1\ , \qquad s^1 = 2j_1\ . \qquad (7{\cdot}27\mathrm{b})$$

The state $||j_1, s^2\rangle$ denotes

$$||j_1, s^2\rangle = (\hat{b}_2^{1*})^{s^1}(\hat{b}^*)^{(2s^2-s^1)/2}|0\rangle\ . \qquad (7{\cdot}28)$$

The quantities j_1, j_2 and j_3 denote

$$j_1,\ j_2,\ j_3 = 0,\ 1/2,\ 1,\ 3/2,\ \cdots . \qquad (7{\cdot}29)$$

We denote the minimum weight state $||j, s^2\rangle$ for the spin j :

$$\hat{J}_-||j, s^2\rangle = 0 \ , \qquad \hat{J}_0||j, s^2\rangle = -j||j, s^2\rangle \ . \tag{7·30}$$

The state $||j_1, s^2\rangle$ shown in Eq.(7·28) is an example of $||j, s^2\rangle$. Except the normalization constant, the state $||j, s^2\rangle$ connects with $||j \pm 1/2, s^2\rangle$ through

$$||j, s^2\rangle = \hat{\boldsymbol{C}}^*_{\mp 1/2}||j \pm 1/2, s^2\rangle \ . \tag{7·31}$$

For the prove of the relation (7·31), we use the relations

$$[\hat{J}_-, \hat{\boldsymbol{C}}^*_{\mp 1/2}] = \hat{\boldsymbol{C}}^*_{\pm 1/2}\hat{J}_- \ , \qquad [\hat{J}_0, \hat{\boldsymbol{C}}^*_{\mp 1/2}] = \pm(1/2)\hat{\boldsymbol{C}}^*_{\mp 1/2} \ . \tag{7·32}$$

The relation (7·31) tells us that the operators $\hat{\boldsymbol{C}}^*_{\mp 1/2}$ play a role of the lowering (the upper sign) and the raising (the lower sign) operator for the state $||j, s^2\rangle$. With the successive use of the relation (7·31), we obtain

$$||j_3, s^2\rangle = (\hat{\boldsymbol{C}}^*_{-1/2})^{-j_3+j_2+j_1}(\hat{\boldsymbol{C}}^*_{+1/2})^{j_3+j_2-j_1}||j_1, s^2\rangle \ . \tag{7·33}$$

Therefore, $|m\mu(2)\rangle$ shown in Eq.(7·24) can be written as

$$|m\mu(2)\rangle = |j_1 j_2; j_3 m_3, s^2\rangle \ . \tag{7·34}$$

The state (7·33) can be rewritten in the form

$$\begin{aligned} ||j_3, s^2\rangle &= (\hat{\boldsymbol{C}}^*_{+1/2})^{j_3+j_2-j_1}(\hat{\boldsymbol{C}}^*_{-1/2})^{-j_3+j_2+j_1}||j_1, s^2\rangle \\ &= (\hat{\boldsymbol{C}}^*_{+1/2})^{j_3+j_2-j_1}||(j_3 - j_2 + j_1)/2, s^2\rangle \ . \end{aligned} \tag{7·35}$$

Since $-j_3 + j_2 + j_1 \geq 0$, $j_3 + j_2 - j_1 \geq 0$ and $j_3 - j_2 + j_1 \geq 0$, we have

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2 \ . \tag{7·36}$$

We can see that the state (7·34) can be regarded as the state

$$|j_1 j_2; j_3 m_3, s^2\rangle = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | j_3 m_3\rangle \hat{\boldsymbol{C}}^*_{j_2 m_2} |j_1 m_1, s^2\rangle \ . \tag{7·37}$$

Here, $\langle j_1 m_1 j_2 m_2 | j_3 m_3\rangle$ denotes the Clebsch-Gordan coefficient and $\hat{\boldsymbol{C}}^*_{j_2 m_2}$ is defined as

$$\hat{\boldsymbol{C}}^*_{j_2 m_2} = \left(\sqrt{(j_2 + m_2)!(j_2 - m_2)!}\right)^{-1} (\hat{\boldsymbol{C}}^*_{+1/2})^{j_2+m_2}(\hat{\boldsymbol{C}}^*_{-1/2})^{j_2-m_2} \ . \tag{7·38}$$

The operator $\hat{\boldsymbol{C}}^*_{j_2 m_2}$ denotes spherical tensor with rank j_2. The state $|j_1 m_1, s^2\rangle$ is given as

$$|j_1 m_1, s^2\rangle = (\hat{J}_+)^{j_1+m_1}||j_1, s^2\rangle \ . \tag{7·39}$$

Finally, we will give short concluding remarks. In this paper, we developed the generalized Lipkin model in the Schwinger boson representation. Not only the symmetric representation but also non-symmetric representation for the $su(M+1)$ algebra was formulated. At the same time, the $su(M,1)$ algebra was also treated. We already knew that the non-compact algebra, for example, the $su(1,1)$ algebra, plays an interesting role for describing various thermal phenomenon, such as non-equilibrium state.[12] We can find the reason why such descriptions are possible in the phase space doubling. The $su(M,1)$ algebra may play the role of the phase space doubling. This is our next problem.

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Appendix A

—— *The eigenvalues of the Casimir operators* ——

First, we investigate the quadratic form $\sum_{c,d=1}^{L} \mathit{\Lambda}_{cd} X_c X_d$. Here, X_c, X_d and $\mathit{\Lambda}_{cd}$ are real and $\mathit{\Lambda}_{cd} = \mathit{\Lambda}_{dc}$. The orthogonal transformation permits us to express the above quadratic form in the following form :

$$\sum_{c,d=1}^{L} \mathit{\Lambda}_{cd} X_c X_d = \sum_{e=1}^{L} \lambda_e (X'_e)^2 \ . \tag{A·1}$$

The quantity λ_e denotes the eigenvalue of the symmetric matrix $(\mathit{\Lambda}_{cd})$. The relation between two vectors (X_c) and (X'_e) are given by

$$X'_e = \sum_{c=1}^{L} y_c^{(e)} X_c \ , \qquad X_c = \sum_{e=1}^{L} y_c^{(e)} X'_e \ . \tag{A·2}$$

Here, $(y_c^{(e)})$ is an orthogonal matrix obeying the conditions

$$\sum_{d=1}^{L} \mathit{\Lambda}_{cd} y_d^{(e)} = \lambda_e y_c^{(e)} \ , \quad (e = 1, 2, \cdots, L) \tag{A·3}$$

$$\sum_{c=1}^{L} y_c^{(e)} y_c^{(f)} = \delta_{ef} \ , \qquad \sum_{e=1}^{L} y_c^{(e)} y_d^{(e)} = \delta_{cd} \ . \tag{A·4}$$

In this Appendix, we will treat the following form :

$$\Lambda_{cd} = \delta_{cd} - (L+1)^{-1} \ . \tag{A·5}$$

It can be easily shown that the eigenvalue equation (A·3) for the form (A·5) has the following solution :

$$\lambda_e = \begin{cases} 1 \ , & (e = 1, 2, \cdots, L-1) \\ (L+1)^{-1} \ . & (e = L) \end{cases} \tag{A·6}$$

For the case $e = L$, $y_c^{(L)}$ is given by

$$y_c^{(L)} = 1/\sqrt{L} \ . \quad (c = 1, 2, \cdots, L) \tag{A·7}$$

However, for the case $e = 1, 2, \cdots, L-1$, all λ_e are equal to 1 and, then, it is impossible to determine $y_c^{(e)}$ uniquely. Only we can show that $y_c^{(e)}$ should obey

$$\begin{aligned} &\sum_{c=1}^{L} y_c^{(e)} = 0 \ , && (e = 1, 2, \cdots, L-1) \\ &\sum_{c=1}^{L} y_c^{(e)} y_c^{(f)} = \delta_{ef} \ . && (e, f = 1, 2, \cdots, L-1) \end{aligned} \tag{A·8}$$

With the use of the above result, the relation (A·1) becomes of the form

$$\sum_{c=1}^{L} (X_c)^2 - (L+1)^{-1} \left(\sum_{c=1}^{L} X_c \right)^2 = \sum_{e=1}^{L-1} (X_e')^2 + (L+1)^{-1} (X_L')^2 \ . \tag{A·9}$$

For the convenience of the discussion in §§3 and 4, we adopt the following form for $y_c^{(e)}$ in $e = 1, 2, \cdots, L-1$:

$$y_c^{(e)} = \begin{cases} -(\sqrt{e(e+1)})^{-1} \ , & (c = 1, 2, \cdots, e) \\ \sqrt{e/(e+1)} \ , & (c = e+1) \\ 0 \ , & (c = e+2, \cdots, L) \end{cases} \tag{A·10}$$

or

$$y_c^{(e)} = \begin{cases} 0 \ , & (e = 1, 2, \cdots, c-2) \\ \sqrt{e/(e+1)} \ , & (e = c-1) \\ -(\sqrt{e(e+1)})^{-1} \ . & (e = c, \cdots, L-1) \end{cases} \tag{A·11}$$

Direct calculation tells us that the above $y_c^{(e)}$ satisfies the relations (A·8) and the relation

$$2 \sum_{c=1}^{L} c y_c^{(e)} = \sqrt{e(e+1)} \ . \quad (e = 1, 2, \cdots, L-1) \tag{A·12}$$

The relation (A·12) leads us to the following form :

$$\sum_{c=1}^{L} (L - 2c) X_c = -\sum_{e=1}^{L-1} \sqrt{e(e+1)} X_e' - \sqrt{L} X_L' \ . \tag{A·13}$$

The form (A·13) is used in §§3 and 4. Thus, we have

$$\sum_{c=1}^{L}(X_c)^2-(L+1)^{-1}\left(\sum_{c=1}^{L}X_c\right)^2\mp\sum_{c=1}^{L}(L-2c)X_c$$
$$=\sum_{e=1}^{L-1}[(X'_e)^2\pm\sqrt{e(e+1)}X'_e]+(L+1)^{-1}(X'_L)^2\pm\sqrt{L}X'_L\ . \qquad \text{(A·14)}$$

For the convenience of the discussion in §§3 and 4, we introduce new parameter X^e ($e=1,2,\cdots,L$) as follows :

$$X^e=\sqrt{(e+1)/e}\ X'_e\ , \quad (e=1,2,\cdots,L-1)\ , \qquad X^L=(1/\sqrt{L})X'_L\ . \qquad \text{(A·15)}$$

Then, the relation (A·14) can be rewritten in the form

$$\sum_{c=1}^{L}(X_c)^2-(L+1)^{-1}\left(\sum_{c=1}^{L}X_c\right)^2\mp\sum_{c=1}^{L}(L-2c)X_c$$
$$=\sum_{e=1}^{L-1}[e/(e+1)]X^e[X^e\pm(e+1)]+[L/(L+1)]X^L[X^L\pm(L+1)]\ , \qquad \text{(A·16)}$$

$$X^e=\begin{cases}-(1/e)\displaystyle\sum_{c=1}^{e}X_c+X_{e+1}\ , & (e=1,2,\cdots,L-1)\\ (1/L)\displaystyle\sum_{c=1}^{L}X_c\ , & (e=L)\end{cases} \qquad \text{(A·17)}$$

or

$$X_c=[(c-1)/c]X^{c-1}-\sum_{e=c}^{L-1}(e+1)^{-1}X^e+X^L\ . \qquad \text{(A·18)}$$

Appendix B

—— *The proof of the properties of $\hat{B}_r^*$* ——

Let us give the proof of the properties of $\hat{B}_r^*$, which is defined in the form (4·4). First, we note the following relations :

$$[\hat{S}_i^j,\hat{b}_i^{s*}]=-\hat{b}_j^{s*}\ , \qquad \text{(B·1a)}$$
$$[\hat{S}_i^j,\hat{b}_k^{s*}]=0\ , \quad (k\neq i) \qquad \text{(B·1b)}$$

$$[\hat{T}_q^p,\hat{b}_k^{q*}]=\hat{b}_k^{p*}\ , \qquad \text{(B·2a)}$$
$$[\hat{T}_q^p,\hat{b}_k^{s*}]=0\ . \quad (s\neq q) \qquad \text{(B·2b)}$$

It should be noted that by calculating the commutation relations (B·1a) and (B·2a), we obtain $\hat{b}_j^{s*}$ and $\hat{b}_k^{p*}$ from $\hat{b}_i^{s*}$ and $\hat{b}_k^{q*}$ by changing i and q with j and p, respectively. For the case $i = 1, 2, \cdots, M-r$, $\hat{B}_r^*$ does not contain $\hat{b}_i^{s*}$ and, then, in this case, we have the relation (4·6a). For the case $i = M-r+1, \cdots, M$, we make the cofactor expansion for the determinant $\hat{B}_r^*$:

$$\hat{B}_r^* = \sum_{s=1}^{r} \hat{b}_i^{s*} \hat{\Delta}_s^{i*} \ . \tag{B·3}$$

Here, $\hat{\Delta}_s^{i*}$ denotes the cofactor of (s, i). Then, the commutation relation $[\hat{S}_i^j, \hat{B}_r^*]$ for $(j > i)$ gives us the following relation :

$$\begin{aligned}[\hat{S}_i^j, \hat{B}_r^*] &= -\sum_{s=1}^{r} \hat{b}_j^{s*} \hat{\Delta}_s^{i*} \\ &= -\begin{vmatrix} \hat{b}_M^{1*} & \cdots & \hat{b}_M^{r*} \\ \cdots & \cdots & \cdots \\ \hat{b}_j^{1*} & \cdots & \hat{b}_j^{r*} \\ \cdots & \cdots & \cdots \\ \hat{b}_j^{1*} & \cdots & \hat{b}_j^{r*} \\ \cdots & \cdots & \cdots \\ \hat{b}_{M-r+1}^{1*} & \cdots & \hat{b}_{M-r+1}^{r*} \end{vmatrix} = 0 \ . \end{aligned} \tag{B·4}$$

The reason is as follows : For the sake of the relation (B·1a), $\hat{b}_i^{s*}$ in $\hat{B}_r^*$ changes to $\hat{b}_j^{s*}$, which exists already in the j-th row $(j > i)$. Therefore, the determinant becomes vanish. Thus, we have the relation (4·6a). The case (4·6b) is also in the same situation as the above. First, we make the cofactor expansion :

$$\hat{B}_r^* = \sum_{k=M-r+1}^{M} \hat{b}_k^{q*} \hat{\Delta}_q^{k*} \ . \tag{B·5}$$

If $q = 1, 2, \cdots, r$ and $q > p$, we have

$$\begin{aligned}[\hat{T}_q^p, \hat{B}_r^*] &= \sum_{k=M-r+1}^{M} \hat{b}_k^{p*} \hat{\Delta}_q^{k*} \\ &= \begin{vmatrix} \hat{b}_M^{1*} & \vdots & \hat{b}_M^{p*} & \vdots & \hat{b}_M^{p*} & \vdots & \hat{b}_M^{r*} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{b}_{M-r+1}^{1*} & \vdots & \hat{b}_{M-r+1}^{p*} & \vdots & \hat{b}_{M-r+1}^{p*} & \vdots & \hat{b}_{M-r+1}^{r*} \end{vmatrix} = 0 \ . \end{aligned} \tag{B·6}$$

The reason why the determinant vanishes is the same as the previous case. In the case $q = r+1, \cdots, M$, $\hat{B}_r^*$ does not contain $\hat{b}_k^{q*}$ in $\hat{B}_r^*$ and, then, the commutation relation $[\hat{T}_q^p, \hat{B}_r^*]$ becomes vanish. Thus, we obtain the relation (4·6b).

Next, let us prove the relations (4·9a) and (4·9b). In the case $i = 1, 2, \cdots, M-r$, $\hat{B}_r^*$ does not contain $\hat{b}_i^{s*}$ and we have $[\hat{S}_i^i, \hat{B}_r^*] = 0$. In the case $i = M-r+1, \cdots, M$, the relation

(B·3) gives us

$$[\hat{S}_i^i, \hat{B}_r^*] = -\begin{vmatrix} \hat{b}_M^{1*} & \cdots & \hat{b}_M^{r*} \\ \cdots & \cdots & \cdots \\ \hat{b}_{M-r+1}^{1*} & \cdots & \hat{b}_{M-r+1}^{r*} \end{vmatrix} = -\hat{B}_r^* \ . \tag{B·7}$$

In this case, $\hat{b}_i^{s*}$ is replaced with $\hat{b}_i^{s*}$ itself. In the case $p = 1, 2, \cdots, r$, we have

$$[\hat{T}_p^p, \hat{B}_r^*] = \begin{vmatrix} \hat{b}_M^{1*} & \vdots & \hat{b}_M^{r*} \\ \vdots & \vdots & \vdots \\ \hat{b}_{M-r+1}^{1*} & \vdots & \hat{b}_{M-r+1}^{r*} \end{vmatrix} = \hat{B}_r^* \ . \tag{B·8}$$

In the case $p = r+1, \cdots, M$, $\hat{B}_r^*$ does not contain $\hat{b}_k^{p*}$ and we have $[\hat{T}_p^p, \hat{B}_r^*] = 0$. By expressing the change of r in terms of i and p, we obtain the relations (4·9a) and (4·9b).

Finally, the proof of the relations (4·14) is sketched. As an example, we will show the case $[\hat{S}^i, \hat{A}_M^*] = 0$. The other relations can be proved in the way similar to the case $[\hat{S}^i, \hat{A}_M^*] = 0$. For the proof of $[\hat{S}^i, \hat{A}_M^*] = 0$, the following relations must be proved :

$$\left[\sum_{p=1}^{M} \hat{a}^{p*}\hat{b}_i^p, \hat{B}_M^*\right] = \hat{B}_M^*(i) \ , \tag{B·9}$$

$$\left[\sum_{p=1}^{M} \hat{a}^{p*}\hat{b}_i^p, \hat{B}_M^*(l)\right] = 0 \ . \tag{B·10}$$

With the use of the expansion (B·3) for $r = M$, we have

$$\begin{aligned}\left[\sum_{p=1}^{M} \hat{a}^{p*}\hat{b}_i^p, \hat{B}_M^*\right] &= \left[\sum_{p=1}^{M} \hat{a}^{p*}\hat{b}_i^p, \sum_{q=1}^{M} \hat{b}_i^{q*}\hat{\Delta}_q^{i*}\right] \\ &= \sum_{q=1}^{M} \hat{a}^{q*}\hat{\Delta}_q^{i*} = \hat{B}_M^*(i) \ . \end{aligned}\tag{B·11}$$

For the relation (B·10), we divide the cases $i = l$ and $i \neq l$. In the case $i = l$, $\hat{B}_M^*(l)$ does not contain $\hat{b}_i^{p*}$ and, then, we have the relation (B·10). In the case $i \neq l$, the commutator becomes of the form that the elements of the i-th row are identical with those of the l-th row $(\hat{a}^{1*}, \cdots, \hat{a}^{M*})$ and the determinant becomes vanish. Then, through the following procedure, we get the relation $[\hat{S}^i, \hat{A}_M^*] = 0$:

$$\begin{aligned}\left[\hat{S}^i, \hat{A}_M^*\right] &= \left[\hat{a}_i^*\hat{b} + \sum_{p=1}^{M} \hat{a}^{p*}\hat{b}_i^p, \hat{B}_M^*\hat{b} + \sum_{l=1}^{M} \hat{B}_M^*(l)\hat{a}_l\right] \\ &= -\sum_{l=1}^{M} \hat{B}_M^*(l)\,[\hat{a}_l, \hat{a}_i^*]\,\hat{b} + \left[\sum_{p=1}^{M} \hat{a}^{p*}\hat{b}_i^p, \hat{B}_M^*\right]\hat{b} + \sum_{l=1}^{M}\left[\sum_{p=1}^{M} \hat{a}^{p*}\hat{b}_i^p, \hat{B}_M^*(l)\right]\hat{a}_l \\ &= -\hat{B}_M^*(i)\hat{b} + \hat{B}_M^*(i)\hat{b} = 0 \ . \end{aligned}\tag{B·12}$$

References

1) H. J. Lipkin, N. Meshkov and A. Glick, Nucl. Phys. **62** (1965), 188.
2) A. Kuriyama, J. da Providência, Y. Tsue and M. Yamamura, Prog. Theor. Phys. **94** (1995), 1039 ; **95** (1996), 339 ; **96** (1996), 125.
   L. Brito, C. Providência and J. da Providência, Mod. Phys. Lett. **A12** (1997), 2985.
3) S. Y. Li, A. Klein and R. M. Dreizler, J. Math. Phys. **11** (1970), 975.
   F. Sakata, Y. Hashimoto, T. Marumori and T. Une, Prog. Theor. Phys. **70** (1983), 163.
   M. Matsuo and K. Matsuyanagi, Prog. Theor. Phys. **74** (1985), 288.
   K. Takada, Y. R. Shimizu and H. Thorn, Nucl. Phys. **A485** (1988), 189.
   M. Sambataro, Phys. Rev **C60** (1999), 0643200.
4) M. O. Terra, M. C. Nemes, C. Providência and J. da Providência, Ann. of Phys. **262** (1998), 1.
5) M. Yamamura and A. Kuriyama, Prog. Theor. Phys. Suppl. **93** (1987).
6) T. Holstein and H. Primakoff, Phys. Rev. **58** (1940), 1098.
   S. C. Pang, A. Klein and R. M. Dreizler, Ann. of Phys. **49** (1968), 477.
7) J. Schwinger, in *Quantum Theory of Angular Momentum*, ed. L. Biedenharn and H. Van Dam (Academic Press, New York, 1955), p.229.
8) A. Klein and E. R. Marshalek, Rev. Mod. Phys. **63** (1991), 375.
9) A. Kuriyama, J. da Providência and M. Yamamura, Prog. Theor. Phys. **103** (2000), 285.
10) M. Moshinsky, *Group Theory and the Many-Body Problem*, (Gordon and Breach, Science Publishers, New York, London, Paris, 1968).
11) A. Kuriyama, J. da Providência and M. Yamamura, Prog. Theor. Phys. **103** (2000), 305.
12) A. Kuriyama, J. da Providência, Y. Tsue and M. Yamamura, Prog. Theor. Phys. Suppl. (to be published).